Arithmetic and Zariski-dense subgroups: weak commensurability, eigenvalue rigidity, and applications to locally symmetric spaces

Andrei S. Rapinchuk
University of Virginia

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Then:

- $u^+(1)$ and $u^-(1)$ generate $\mathrm{SL}_2(\mathbb{Z})$ which is arithmetic;
- $u^+(2)$ and $u^-(2)$ generate a subgroup of index 12 $\mathrm{SL}_2(\mathbb{Z})$, which is again arithmetic;
- for $m \geqslant 3$, $u^+(m)$ and $u^-(m)$ generate a Zariski-dense subgroup of infinite index in $\mathrm{SL}_2(\mathbb{Z})$, which is not arithmetic (*thin*)

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Here a subset $\Phi \subset \Gamma$ is called *free* if inclusion $\Phi \hookrightarrow \Gamma$ extends to *injective* homomorphism of free group on Φ to Γ.

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We would like to extend some notions from arithmetic groups to arbitrary Zariski-dense subgroups.

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We let K_Γ denote the trace field of Γ, i.e., subfield of $\mathbb{C}$ generated by

$$\mathrm{Tr}\,\mathrm{Ad}_G(\gamma), \quad \gamma \in \Gamma.$$

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- If G is simple and Γ is $(K, \mathcal{G})$-arithmetic, then

$$K_\Gamma = K \text{ and } G^0(\Gamma) = \mathcal{G}.$$

Thus, K_Γ and $G^0(\Gamma)$ are *analogs* of K and $\mathcal{G}$ for arbitrary Zariski-dense subgroups.

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But it is as close an analogy as it can only be.

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(1) Let $\gamma_1 \in \mathrm{GL}_{n_1}(F)$ and $\gamma_2 \in \mathrm{GL}_{n_2}(F)$ be *semi-simple* matrices, let

$$\lambda_1, \ldots, \lambda_{n_1} \quad \text{and} \quad \mu_1, \ldots, \mu_{n_2} \quad (\in \overline{F})$$

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be their eigenvalues. Then γ_1 and γ_2 are *weakly commensurable* if $\exists\ a_1, \ldots, a_{n_1}\ ,\ b_1, \ldots, b_{n_2} \in \mathbb{Z}$ such that

$$\lambda_1^{a_1} \cdots \lambda_{n_1}^{a_{n_1}} = \mu_1^{b_1} \cdots \mu_{n_2}^{b_{n_2}} \neq 1.$$

Let $G_1 \subset \mathrm{GL}_{n_1}$ and $G_2 \subset \mathrm{GL}_{n_2}$ be reductive F-groups, $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ be Zariski-dense subgroups.

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(2) Γ_1 and Γ_2 are *weakly commensurable* if

every semi-simple $\gamma_1 \in \Gamma_1$ of infinite order

is weakly commensurable to

some semi-simple $\gamma_2 \in \Gamma_2$ of infinite order,

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$\overset{(1)}{\Leftrightarrow}$ there exists maximal F-tori T_i of G_i such that $\gamma_i \in T_i(F)$ and characters $\chi_i \in X(T_i)$ $(i = 1, 2)$ for which

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Remark. These reformulations show that weak commensurability is *independent* of matrix realizations of G_i's.

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Theorem 1

If *Γ_1 and Γ_2 are weakly commensurable,* **then** *either G_1 and G_2 have same Killing-Cartan type, or one of them is of type B_ℓ and the other of type C_ℓ $(\ell \geqslant 3)$.*

For a Zariski-dense subgroup $\Gamma \subset G(F)$, let

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$$G^0(\Gamma_1) \simeq G^0(\Gamma_2) \text{ over } K \Rightarrow \Gamma_1 \,\&\, \Gamma_2 \text{ commensurable.}$$

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Finiteness conjecture.

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- G_1 *and* G_2 *be absolutely simple algebraic* F*-groups,* $\mathrm{char}\, F = 0$*;*
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There exists a finite collection $\mathcal{G}_2^{(1)}, \ldots, \mathcal{G}_2^{(r)}$ *of F/K-forms of* G_2 *such that* **if**

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(Additionally, one expects that $r = 1$ in certain situations ...)

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FINITENESS CONJECTURE $\Rightarrow$ There are only finitely many c.s.a. A' such that for $G' = \mathrm{PSL}_{1,A'}$,

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- Similar consequences for orthogonal groups of quadratic forms etc.

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General case is work in progress ...

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Let

- G_1 *and* G_2 *be absolutely almost simple F-groups,* $\mathrm{char}\, F = 0$*;*
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Remark. Types excluded in (1) are *honest exceptions*.

(cont.)

(3) **If** Γ_1 *and* Γ_2 *are weakly commensurable, and* $K = K_{\Gamma_1} = K_{\Gamma_2}$, **then** $\mathrm{rk}_K\, G^0(\Gamma_1) = \mathrm{rk}_K\, G^0(\Gamma_2)$.

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Now, let G_1 and G_2 be *absolutely almost simple* $\mathbb{R}$-groups,

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Proposition

Assume that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ have finite volume (i.e., Γ_1 and Γ_2 are lattices).

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Conjecture (Shanuel) *If* $z_1, \ldots, z_n \in \mathbb{C}$ *are linearly independent over* $\mathbb{Q}$*, then the transcendence degree of field generated by*

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A finite volume locally symmetric space $\mathfrak{X}_\Gamma$ of a simple real group is automatically *arithmetically defined* unless $\mathfrak{X}$ is either real hyperbolic space $\mathbb{H}^n$ or complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$.

(Margulis + Corlette + Gromov-Schoen)

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- *The set of* $\mathfrak{X}_{\Gamma_2}$*'s length-commensurable to* $\mathfrak{X}_{\Gamma_1}$ *is a union of finitely many commensurability classes.*

It consists of single commensurability class if G_1 *and* G_2 *are of same type different from* A_n*,* D_{2n+1} $(n > 1)$*, or* E_6*.*

Corollary

Let M_1 *and* M_2 *be arithmetically defined hyperbolic d-manifolds, where* $d \neq 3$ *is even or* $\equiv 3 (\mathrm{mod}\, 4)$.

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So, $L(M_i)$ contains "many" elements that are algebraically independent from *all* elements of $L(M_{3-i})$.

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Assume that both G_1 *and* G_2 *are of one of following types:* A_n, D_{2n+1} $(n > 1)$ *or* E_6, *subgroups* Γ_1 *and* Γ_2 *are arithmetic, and in addition* $K_{\Gamma_i} \neq \mathbb{Q}$ *for at least one* $i \in \{1, 2\}$.

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Assume that G_1 and G_2 are either of same type or one of them is of type B_ℓ and other of type C_ℓ, and let $M_i = \mathfrak{X}_{\Gamma_i}$ $(i = 1, 2)$ be arithmetically defined locally symmetric spaces.

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Results for isospectral locally symmetric spaces are derived from those for length-commensurable spaces.

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Example. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and set

$$u^+(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \text{ and } u^-(b) = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}.$$

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Then for $m \geqslant 3$, subgroup

$$\Delta_m := \langle u^+(m)\,, u^-(m) \rangle$$

is of infinite index in Γ, **but** is weakly commensurable to it.

Weak commensurability follows from inclusion

$$\Gamma(m^2) \subset \bigcup_{g \in \mathrm{GL}_2(\mathbb{Q})} g\,\Delta_m\,g^{-1},$$

where

$$\Gamma(m^2) = \{\, x \in \Gamma \mid x \equiv I_2 \pmod{m^2} \,\}$$

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So, we would like to propose the following

Problem 1. Let G_1 and G_2 be simple algebraic groups over a field F of characteristic zero, and let $\Gamma_1 \subset G_1(F)$ be an arithmetic subgroups of rank $\geqslant 2$.

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Problem can be stated for higher-rank S-arithmetic subgroups, but is wide-open even for $\mathrm{SL}_2(\mathbb{Z}[1/p])$.

Problem 2. Let G_1 and G_2 be simple groups over $F = \mathbb{R}$ or $\mathbb{C}$, and let Γ_i be a (finitely generated) Zariski-dense subgroup of $G_i(F)$ for $i = 1, 2$. Assume that Γ_1 and Γ_2 are weakly commensurable.

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Does discreteness of Γ_1 imply discreteness of Γ_2?

The answer is 'yes' for a nonarchimedean locally compact field F, but archimedean case is open.

Problem 3. Let G_1 and G_2 be simple algebraic groups over $F = \mathbb{R}$ or $\mathbb{C}$, and let $\Gamma_i \subset G_i(F)$ be a lattice for $i = 1, 2$. Assume that Γ_1 and Γ_2 are weakly commensurable.

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Does compactness of $\mathfrak{X}_{\Gamma_1}$ imply compactness of $\mathfrak{X}_{\Gamma_2}$?

Recall: The answer is 'yes' if one space is arithmetically defined.

Problem 4. Develop notion of weak commensurability for Zariski-dense (and particularly arithmetic) subgroups of general semi-simple groups.

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Problem 5. For inner and outer forms of types A_n $(n > 1)$, D_{2n+1} $(n > 1)$ and E_6, construct examples of isospectral compact arithmetically defined locally symmetric spaces that are not commensurable.

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Currently, such construction is available only for inner forms of type A_n.

1. Algebraic groups and their arithmetic and Zariski-dense subgroups
   - Zariski-dense subgroups
2. Results
   - First signs of eigenvalue rigidity
   - Weakly commensurable arithmetic groups
   - Geometric applications
3. Some open problems
4. **Generic elements**
5. Nonarithmetic Riemann surfaces

Let $A \in \mathrm{GL}_n(F)$, and let $\chi_A(t)$ = characteristic polynomial of A.

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We will now generalize notion of generic elements and existence theorem to arbitrary semi-simple groups.

Let G be a semi-simple algebraic group over a field F, let T be a maximal F-torus, and $\Phi = \Phi(G, T)$ corresponding root system.

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Recall: action of $\mathcal{G} = \mathrm{Gal}(\overline{F}/F)$ on character group $X(T)$ gives rise to group homomorphism

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Definition.

(1) T is generic over F if $\mathrm{Im}\, \theta_T$ contains Weyl group $W(\Phi)$.

(2) A semi-simple element $\gamma \in G(F)$ is generic over F if $T := Z_G(\gamma)^\circ$ is a torus (i.e., γ is *regular*) which is generic over F.

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Let G be a semi-simple algebraic group over a finitely generated field F, let $\Gamma \subset G(F)$ be a finitely generated Zariski-dense subgroup.

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1. *Γ contains an F-generic element $\gamma \in \Gamma$ without components of finite order;*
2. *if $\gamma \in \Gamma$ is F-generic then there exists a finite index subgroup $\Delta \subset \Gamma$ such that $\gamma\Delta$ consists of F-generic elements.*

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Remarks. "Components" in (1) refer to almost direct product $G = G_1 \cdots G_r$ of simple groups.
(2) means that set of F-regular elements is open in Γ for profinite topology.

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Such elements were used to study dynamics of actions, rigidity, Auslander problem about properly discontinuous groups of affine transformations, etc.

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This is false for dense subgroups of compact tori!

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- **If** Γ is *arithmetic*, **then** A_Γ is the quaternion algebra involved in its description;
- In general, A_Γ **does not** determine Γ, **but** is an invariant of the commensurability class of Γ.

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  if $\gamma \sim \pm \begin{pmatrix} t_\gamma & 0 \\ 0 & t_\gamma^{-1} \end{pmatrix}$ $(t_\gamma > 1)$ then *length* $\ell(c_\gamma) = 2\log t_\gamma$;
- *algebraically*: a maximal etale subalgebra $K_\Gamma[\gamma] \subset A_\Gamma$.

Now, let $M_i = \mathbb{H}/\Gamma_i$ $(i = 1, 2)$ be length-commensurable.

Then:

1. $K_{\Gamma_1} = K_{\Gamma_2} =: K$;
2. Given closed geodesics $c_{\gamma_i} \subset M_i$ for $i = 1, 2$ such that
$$\ell(c_{\gamma_2})/\ell(c_{\gamma_1}) = m/n \quad (m, n \in \mathbb{Z})$$
elements γ_1^m and γ_2^n are conjugate $\Rightarrow$

   $K[\gamma_1] \subset A_{\Gamma_1}$ and $K[\gamma_2] \subset A_{\Gamma_2}$ are isomorphic.

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What can one say about quaternion algebras associated to length-commensurable Riemann surfaces?

- For M_1 and M_2 to be commensurable, A_{Γ_1} and A_{Γ_2} must be isomorphic.

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Theorem 10

Let $M_i = \mathbb{H}/\Gamma_i$ $(i \in I)$ be a family of length-commensurable *Riemann surfaces where $\Gamma_i \subset \mathrm{PSL}_2(\mathbb{R})$ is finitely generated and Zariski- dense.* **Then** *quaternion algebras A_{Γ_i} $(i \in I)$ split into finitely many isomorphism classes (over common center).*