Arithmetic and Zariski-dense subgroups:

weak commensurability, eigenvalue rigidity, and applications to locally symmetric spaces

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Krakow July, 2018

1. Algebraic groups and their arithmetic and Zariski-dense subgroups
   - Basic results about arithmetic groups
   - Arithmetic lattices in simple Lie groups
   - Zariski-dense subgroups

2. Results
   - First signs of eigenvalue rigidity
   - Weakly commensurable arithmetic groups
   - Geometric applications

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(2) $G(\mathbb{R})/G(\mathbb{Z})$ *is compact $\Leftrightarrow$ G is $\mathbb{Q}$-anisotropic.*

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Definition.

An abstract group Γ is said to have bounded generation if there exist $\gamma_1, \ldots, \gamma_d$ such that

$$\Gamma = \langle \gamma_1 \rangle \cdots \langle \gamma_d \rangle,$$

where $\langle \gamma_i \rangle$ is the cyclic group generated by γ_i.

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2. Every m-dimensional $\mathbb{Q}$-torus is a $\mathbb{C}/\mathbb{Q}$-form of $\mathbb{D}_m$.

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a subgroup $\Gamma \subset \mathcal{G}$ *is* arithmetic *if there is* $\mathbb{R}/\mathbb{Q}$*-form* G' *of* G **and** $\mathbb{R}$*-isomorphism* $\varphi \colon G' \to G$ *such that* Γ *is commensurable with* $\varphi(G'(\mathbb{Z}))$.

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Then each of the *rational* quadratic forms

$$q_1 = x^2 + y^2 - 3z^2 \text{ and } q_2 = x^2 + y^2 - 7z^2,$$

being equivalent to q over $\mathbb{R}$, defines a family of arithmetic subgroups of $\mathcal{G}$.

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So, $\Gamma = \mathrm{SO}_3(q_3)(\mathbb{Z}[\sqrt{2}])$ embeds as a *discrete* subgroup in

$$\mathcal{H} = \mathcal{G}_3 \times \mathcal{G}_3'$$

where $\mathcal{G}_3 = \mathrm{SO}_3(q_3)(\mathbb{R})$, $\mathcal{G}_3' = \mathrm{SO}_3(q_3')$, $q_3' = x^2 + y^2 + \sqrt{2}z^2$.

By *restriction of scalars*, one constructs a semi-simple $\mathbb{Q}$-group H such that

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So, a "reasonable definition" of an arithmetic group/lattice must include groups that arise from rings of algebraic integers other than $\mathbb{Z}$.

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If G is not adjoint and $\pi \colon G \to \overline{G}$ is F-isogeny onto adjoint group, then $\Gamma \subset G(F)$ is $(K, \mathcal{G})$-arithmetic if $\pi(\Gamma) \subset \overline{G}(F)$ is such.

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Γ_1 and Γ_2 are commensurable up to F-isomorphism between G_1 and G_2 if there exists F-isomorphism

$$\varphi\colon G_1 \to G_2$$

such that $\varphi(\Gamma_1)$ and Γ_2 are commensurable in usual sense.

If G_1 and G_2 are not necessarily adjoint, we consider isogenies

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onto corresponding adjoint groups, and say that subgroups $\Gamma_i \subset G_i(F)$ are commensurable up to F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$ if $\pi_1(\Gamma_1)$ and $\pi_2(\Gamma_2)$ are such.

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Then *Γ_1 and Γ_2 are commensurable up F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$ $\Leftrightarrow$ $K_1 = K_2 =: K$ and $\mathcal{G}_1$ and $\mathcal{G}_2$ are K-isomorphic.*

So,

$$(\mathbb{Q}\,,\,\mathrm{SO}_3(q_1)) \quad , \quad (\mathbb{Q}\,,\,\mathrm{SO}_3(q_2)) \quad \text{and} \quad (\mathbb{Q}(\sqrt{2})\,,\,\mathrm{SO}_3(q_3))$$

define pairwise noncommensurable arithmetic subgroups (lattices) of $\mathcal{G} = \mathrm{SO}_3(q)(\mathbb{R})$.

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This is part of eigenvalue rigidity.

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This multiplication makes D into a central associative F-algebra, denoted $\left(\dfrac{a,b}{F}\right)$ and called algebra of generalized quaternions.

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This multiplication makes D into a central associative F-algebra, denoted $\left(\frac{a, b}{F}\right)$ and called algebra of generalized quaternions.

Algebra of (usual) Hamiltonian quaternions is $\mathbb{H} = \left(\frac{-1, -1}{\mathbb{R}}\right)$.

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It follows that

$$G = \{ z = z_0 + z_1 i + z_2 j + z_3 k \in D \otimes_F \mathbb{C} \mid z_0^2 - az_1^2 - bz_2^2 + abz_3^2 = 1 \}$$

is a group for quaternionic multiplication.

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It is well-known that $D \otimes_F \mathbb{C} \simeq M_2(\mathbb{C})$. Explicit embedding:

$$\iota \colon D \hookrightarrow M_2(\mathbb{C}), \quad z = z_0 + z_1 i + z_2 j + z_3 k \mapsto \begin{pmatrix} z_0 + z_1 i & b(z_2 + z_3 i) \\ z_2 - z_3 i & z_0 - z_1 i \end{pmatrix},$$

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  Then for $G = \mathrm{SL}_{1,D}$ there exists $\mathbb{R}$-isomorphism

$$\varphi \colon G \to \mathrm{SL}_2.$$

- May assume that $a, b \in \mathcal{O}_K = \mathcal{O}$ (ring of integers), then
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- Such subgroup is discrete (i.e., an arithmetic lattice) $\Leftrightarrow$ for any nonidentity embedding $\epsilon \colon K \hookrightarrow \mathbb{C}$ we have $\epsilon(K) \subset \mathbb{R}$ (in particular, K is totally real) and $D \otimes_{K,\epsilon} \mathbb{R}$ is a division algebra.

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Then:

- $u^+(1)$ and $u^-(1)$ generate $SL_2(\mathbb{Z})$ which is arithmetic;
- $u^+(2)$ and $u^-(2)$ generate a subgroup of index 12 $SL_2(\mathbb{Z})$, which is again arithmetic;
- for $m \geqslant 3$, $u^+(m)$ and $u^-(m)$ generate a Zariski-dense subgroup of infinite index in $SL_2(\mathbb{Z})$, which is not arithmetic (*thin*)

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Let G be a (nontrivial) semi-simple over a field F of characteristic zero, and let $\Gamma \subset G(F)$ be a Zariski-dense subgroup.

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Here a subset $\Phi \subset \Gamma$ is called *free* if inclusion $\Phi \hookrightarrow \Gamma$ extends to *injective* homomorphism of free group on Φ to Γ.

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We would like to extend some notions from arithmetic groups to arbitrary Zariski-dense subgroups.

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Let V be a vector space over a field F, and let $\Gamma \subset \mathrm{GL}(V)$ be a subgroup.

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We let K_Γ denote the trace field of Γ, i.e., subfield of $\mathbb{C}$ generated by

$$\mathrm{Tr}\,\mathrm{Ad}_G(\gamma), \quad \gamma \in \Gamma.$$

Theorem (E.B. Vinberg)

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- If G is simple and Γ is $(K, \mathcal{G})$-arithmetic, then

$$K_\Gamma = K \text{ and } G^0(\Gamma) = \mathcal{G}.$$

Thus, K_Γ and $G^0(\Gamma)$ are *analogs* of K and $\mathcal{G}$ for arbitrary Zariski-dense subgroups.

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But it is as close an analogy as it can only be.

1. Algebraic groups and their arithmetic and Zariski-dense subgroups
   - Basic results about arithmetic groups
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2. Results
   - First signs of eigenvalue rigidity
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(1) Let $\gamma_1 \in \mathrm{GL}_{n_1}(F)$ and $\gamma_2 \in \mathrm{GL}_{n_2}(F)$ be *semi-simple* matrices, let

$$\lambda_1, \ldots, \lambda_{n_1} \quad \text{and} \quad \mu_1, \ldots, \mu_{n_2} \quad (\in \overline{F})$$

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be their eigenvalues. Then γ_1 and γ_2 are *weakly commensurable* if $\exists\ a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2} \in \mathbb{Z}$ such that

$$\lambda_1^{a_1} \cdots \lambda_{n_1}^{a_{n_1}} = \mu_1^{b_1} \cdots \mu_{n_2}^{b_{n_2}} \neq 1.$$

Let $G_1 \subset \mathrm{GL}_{n_1}$ and $G_2 \subset \mathrm{GL}_{n_2}$ be reductive F-groups, $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ be Zariski-dense subgroups.

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(2) Γ_1 and Γ_2 are *weakly commensurable* if

every semi-simple $\gamma_1 \in \Gamma_1$ of infinite order

is weakly commensurable to

some semi-simple $\gamma_2 \in \Gamma_2$ of infinite order,

and vice versa.

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$\stackrel{(1)}{\Leftrightarrow}$ there exists maximal F-tori T_i of G_i such that $\gamma_i \in T_i(F)$ and characters $\chi_i \in X(T_i)$ $(i = 1, 2)$ for which

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$\overset{(2)}{\Leftrightarrow}$ there exist F-defined representations

$$\rho_1 \colon G_1 \longrightarrow \mathrm{GL}_{m_1} \quad \text{and} \quad \rho_2 \colon G_2 \longrightarrow \mathrm{GL}_{m_2}$$

such that $\rho_1(\gamma_1)$ and $\rho_2(\gamma_2)$ have a *nontrivial* common eigenvalue.

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$\overset{(1)}{\Leftrightarrow}$ there exists maximal F-tori T_i of G_i such that $\gamma_i \in T_i(F)$ and characters $\chi_i \in X(T_i)$ $(i = 1, 2)$ for which

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$\overset{(2)}{\Leftrightarrow}$ there exist F-defined representations

$$\rho_1 \colon G_1 \longrightarrow \mathrm{GL}_{m_1} \quad \text{and} \quad \rho_2 \colon G_2 \longrightarrow \mathrm{GL}_{m_2}$$

such that $\rho_1(\gamma_1)$ and $\rho_2(\gamma_2)$ have a *nontrivial* common eigenvalue.

Remark. These reformulations show that weak commensurability is *independent* of matrix realizations of G_i's.

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Theorem 1

If Γ_1 *and* Γ_2 *are weakly commensurable,* **then** *either* G_1 *and* G_2 *have same Killing-Cartan type, or one of them is of type* B_ℓ *and the other of type* C_ℓ $(\ell \geqslant 3)$.

For a Zariski-dense subgroup $\Gamma \subset G(F)$, let

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Recall: If Γ_1 and Γ_2 are *arithmetic* then
$$G^0(\Gamma_1) \simeq G^0(\Gamma_2) \text{ over } K \Rightarrow \Gamma_1 \,\&\, \Gamma_2 \text{ commensurable.}$$

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(Additionally, one expects that $r = 1$ in certain situations ...)

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• Similar consequences for orthogonal groups of quadratic forms etc.

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General case is work in progress ...

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(1) *Assume* G_1 *and* G_2 *are of* **same type***, different from* A_n*,* D_{2n+1} $(n > 1)$*, and* E_6.

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Remark. Types excluded in (1) are *honest exceptions*.

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(3) **If** Γ_1 *and* Γ_2 *are weakly commensurable, and* $K = K_{\Gamma_1} = K_{\Gamma_2}$,
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Remark. Above results were proved in a more general context of S-arithmetic subgroups. (4) is valid for S-arithmetic lattices over any locally compact field F.

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$$\frac{\log \alpha}{\log \beta}$$

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Conjecture (Shanuel) *If* $z_1, \ldots, z_n \in \mathbb{C}$ *are linearly independent over* $\mathbb{Q}$*, then the transcendence degree of field generated by*

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A finite volume locally symmetric space $\mathfrak{X}_\Gamma$ of a simple real group is automatically *arithmetically defined* unless $\mathfrak{X}$ is either real hyperbolic space $\mathbb{H}^n$ or complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$.

(Margulis + Corlette + Gromov-Shoen)

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- *The set of* $\mathfrak{X}_{\Gamma_2}$*'s length-commensurable to* $\mathfrak{X}_{\Gamma_1}$ *is a union of finitely many commensurability classes.*

It consists of single commensurability class if G_1 *and* G_2 *are of same type different from* A_n, D_{2n+1} $(n > 1)$, *or* E_6.

Corollary

Let M_1 *and* M_2 *be arithmetically defined hyperbolic d-manifolds, where* $d \neq 3$ *is even or* $\equiv 3 (\mathrm{mod}\, 4)$.

If M_1 *and* M_2 *are length-commensurable,* **then** *they are commensurable.*

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- A *complex* hyperbolic manifold cannot be length-commensurable to a *real* or *quaternionic* hyperbolic manifold, etc.

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Then *either $M_1 = \mathfrak{X}_{\Gamma_1}$ and $M_2 = \mathfrak{X}_{\Gamma_2}$ are commensurable (hence length-commensurable), or (T_i) and (N_i) hold for at least one $i \in \{1, 2\}$.*

Theorem 7

Assume that both G_1 and G_2 are of one of following types: A_n, D_{2n+1} $(n > 1)$ or E_6, subgroups Γ_1 and Γ_2 are arithmetic, and in addition $K_{\Gamma_i} \neq \mathbb{Q}$ for at least one $i \in \{1, 2\}$.

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Let M_i $(i = 1, 2)$ *be quotients of real hyperbolic space* $\mathbb{H}^{d_i}$ *with* $d_i \neq 3$ *by a torsion free discrete subgroup* Γ_i *of* $G_i(\mathbb{R})$ *where* $G_i = \mathrm{PSO}(d_i, 1)$.

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(1) *If* $d_1 > d_2$ *then* (T_1) *and* (N_1) *hold.*

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(3) *If* $d \equiv 1 \pmod 4$ *and in addition* $K_{\Gamma_i} \neq \mathbb{Q}$ *for at least one* $i \in \{1, 2\}$ *then either* M_1 *and* M_2 *are length-commensurable (although not necessarily commensurable), or* (T_i) *and* (N_i) *hold for at least one* $i \in \{1, 2\}$.

Theorem 8

Assume that G_1 and G_2 are either of same type or one of them is of type B_ℓ and other of type C_ℓ, and let $M_i = \mathfrak{X}_{\Gamma_i}$ $(i = 1, 2)$ be arithmetically defined locally symmetric spaces.

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Results for isospectral locally symmetric spaces are derived from those for length-commensurable spaces.