Lecture 5. Localisation Techniques and Functional Equations

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Isolated fixed points

M a T^k-manifold of dimension $2n$ with a T^k-invariant stably tangentially complex structure

$$c_{\mathcal{T}} \colon \mathcal{T}M \oplus \underline{\mathbb{R}}^{2(l-n)} \to \xi.$$

Assume that fixed points are *isolated*, i. e. the fixed point set M^T is finite.

$p \in M^T$ a fixed point. Have a representation $\mathfrak{r}_p \colon T^k \to GL(l, \mathbb{C})$ in the fibre ξ_p. Then $\xi_p = \mathfrak{r}_1 \oplus \cdots \oplus \mathfrak{r}_n \oplus V$, where each $\mathfrak{r}_i$ is a non-trivial one-dimensional complex T^k-representations, and V is trivial.
In the corresponding coordinates $(z_1, \ldots, z_n, v)$, an element $\boldsymbol{t} = (e^{2\pi i \varphi_1}, \ldots, e^{2\pi i \varphi_k}) \in T^k$ acts by

$$\boldsymbol{t} \cdot (z_1, \ldots, z_n, v) = (e^{2\pi i \langle \boldsymbol{w}_1, \varphi \rangle} z_1, \ldots, e^{2\pi i \langle \boldsymbol{w}_n, \varphi \rangle} z_n, v),$$

where $\varphi = (\varphi_1, \ldots, \varphi_k) \in \mathbb{R}^k$ and $\boldsymbol{w}_j \in \mathrm{Hom}(T^k, S^1) \cong \mathbb{Z}^k$, $1 \leqslant j \leqslant n$, are the weights of the representation $\mathfrak{r}_p$.

The isomorphism $c_{\mathcal{T},p}\colon \mathcal{T}_pM \oplus \underline{\mathbb{R}}^{2(l-n)} \to \xi_p$ induces an orientation of the tangent space $\mathcal{T}_pM$, as both $\underline{\mathbb{R}}^{2(l-n)}$ and $\xi_p \cong \mathbb{C}^l$ are canonically oriented.

Definition

For any fixed point $p \in M^T$, the sign $\sigma(p)$ is $+1$ if the isomorphism

$$\mathcal{T}_pM \xrightarrow{\mathrm{id}\,\oplus\, 0} \mathcal{T}_pM \oplus \mathbb{R}^{2(l-n)} \xrightarrow{c_{\mathcal{T},x}} \xi_p = \mathfrak{r}_1 \oplus \cdots \oplus \mathfrak{r}_n \oplus V \xrightarrow{\mathrm{pr}} \mathfrak{r}_1 \oplus \cdots \oplus \mathfrak{r}_n$$

respects the canonical orientations, and -1 if it does not.

If M is an almost complex T^k-manifold (i.e. $l = n$) then $\mathcal{T}_pM = \mathfrak{r}_1 \oplus \cdots \oplus \mathfrak{r}_n$ and $\sigma(p) = 1$ for every fixed point p.

We refer to

$$\{\boldsymbol{w}_j(p),\ \sigma(p)\colon \quad p \in M^T,\ 1 \leqslant j \leqslant n\}$$

as the fixed point data of $(M, c_{\mathcal{T}})$.

Each integer vector $\boldsymbol{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k$ determines a line bundle

$$\bar{\eta}^{\boldsymbol{n}} := \bar{\eta}_1^{n_1} \otimes \cdots \otimes \bar{\eta}_k^{n_k}$$

over $BT^k = (\mathbb{C}P^\infty)^k$, where η_j is the tautological bundle over the jth factor. Let

$$[\boldsymbol{n}](\boldsymbol{u}) := c_1^U(\bar{\eta}^{\boldsymbol{n}})$$

denote the cobordism first Chern class of $\bar{\eta}^{\boldsymbol{n}}$.
It is given by the power series

$$[\boldsymbol{n}](\boldsymbol{u}) = F_U(\underbrace{u_1, \ldots, u_1}_{n_1}, \ldots, \underbrace{u_k, \ldots, u_k}_{n_k}) \in U^2(BT^k),$$

where $F_U(u_1, \ldots, u_k) = F_U(\cdots F_U(F_U(u_1, u_2), u_3), \ldots, u_k)$ is the iterated substitution in the formal group law of geometric cobordisms. We have

$$[\boldsymbol{n}](\boldsymbol{u}) \equiv \langle \boldsymbol{n}, \boldsymbol{u} \rangle = n_1 u_1 + \cdots + n_k u_k \qquad \text{modulo decomposables.}$$

Localisation formulae

$(M, c_{\mathcal{T}})$ with the fixed point data $\{\boldsymbol{w}_j(p),\ \sigma(p)\colon p \in M^T,\ 1 \leqslant j \leqslant n\}$.

The universal toric genus $\Phi\colon \Omega_{U:T^k} \longrightarrow U^*(BT^k) = \Omega_U[[u_1, \ldots, u_k]]$.

Theorem

For any stably tangentially complex $2n$-dimensional T^k-manifold M with isolated fixed points M^T, the equation

$$\Phi(M) \ = \ \sum_{p \in M^T} \sigma(p) \prod_{j=1}^{n} \frac{1}{[\boldsymbol{w}_j(p)](\boldsymbol{u})}$$

is satisfied in $U^{-2n}(BT^k)$.

The summands on the right hand side of formally belong to the localised ring $S^{-1}U^*(BT^k)$ where S is the set of equivariant Euler classes of nontrivial representations of T^k.

Let $\varphi\colon \Omega_U \to R$ be a genus corresponding to $f(x) = x + \cdots \in R \otimes \mathbb{Q}[[x]]$. The universal localisation theorem may be adapted to express $\varphi(M)$ in terms of the fixed point data:

Theorem

Let $\varphi\colon \Omega_U \to R$ be a genus with torsion-free R, and let M be a stably tangentially complex $2n$-dimensional T^k-manifold with isolated fixed points M^T. Then the equivariant genus $\varphi^T(M) = \varphi(M) + \cdots$ is given by

$$\varphi^T(M) \;=\; \sum_{p \in M^T} \sigma(p) \prod_{j=1}^{n} \frac{1}{f\big(\langle \boldsymbol{w}_j(p), \boldsymbol{x}\rangle\big)},$$

where $\langle \boldsymbol{w}, \boldsymbol{x}\rangle = w_1 x_1 + \cdots + w_k x_k$ for $\boldsymbol{w} = (w_1, \ldots, w_k)$.

Proof. By definition, $\varphi^T = h_\varphi \cdot \Phi$ where $h_\varphi(u_i) = f(x_i)$. Now $f(x)$ is the exponential series of the f.g.l. φF_U, i. e.
$\varphi F_U(u_1, u_2) = f(f^{-1}(u_1) + f^{-1}(u_2))$, hence, $h_\varphi F_U(u_1, u_2) = f(x_1 + x_2)$.
It follows that $h_\varphi\big([\boldsymbol{w}_j(p)](\boldsymbol{u})\big) = f(\langle \boldsymbol{w}_j(p), \boldsymbol{x}\rangle)$. □

Example

The augmentation genus $\varepsilon\colon \Omega_U \to \mathbb{Z}$ corresponds to the series $f(x) = x$; it vanishes on any M^{2n} with $n > 0$. The localisation formula gives

$$\sum_{p \in M^T} \sigma(p) \prod_{j=1}^{n} \frac{1}{\langle \boldsymbol{w}_j(p), \boldsymbol{x} \rangle} \;=\; 0.$$

Let $M = \mathbb{C}P^n$ on which T^{n+1} acts homogeneous coordinatewise. There are $n+1$ fixed points $p_0, \ldots, p_n$, each having a single nonzero coordinate. So $\boldsymbol{w}_j(p_k) = \boldsymbol{e}_j - \boldsymbol{e}_k$ for $0 \leqslant j \leqslant n$, $j \neq k$, and every $\sigma(p_k)$ is positive. We obtain the classical identity

$$\sum_{k=0}^{n} \prod_{j \neq k} \frac{1}{x_j - x_k} \;=\; 0\,.$$

Example

The universal toric genus of $\mathbb{C}P^1$ (with the standard S^1-action) is given by

$$\Phi(\mathbb{C}P^1) = \frac{1}{u} + \frac{1}{\bar{u}}$$

in $U^{-2}(\mathbb{C}P^\infty)$ where $\bar{u} = [-1](u)$ is the inverse in the cobordism f.g.l.

A genus $\varphi\colon \Omega_U \to R$ is rigid on $\mathbb{C}P^1$ iff its defining series $f(x)$ satisfies

$$\frac{1}{f(x)} + \frac{1}{f(-x)} = c$$

in $R \otimes \mathbb{Q}[[x]]$. The general analytic solution is given by

$$f(x) = \frac{x}{b(x^2) + cx/2}, \qquad \text{where} \quad b(0) = 1.$$

In particular, the Todd genus td corresponding to $f(x) = 1 - e^{-x}$ satisfies the equation with $c = 1$. So td is T-rigid on $\mathbb{C}P^1$. In fact td is fibre multiplicative with respect to $\mathbb{C}P^1$ by a result of Hirzebruch.

Example

We can also consider the S^1-action on $M = \mathbb{C}P^1$ with trivial stably complex structure.
It has two fixed points of signs 1 and -1, both with weights 1. Then

$$\Phi(M) = \frac{1}{u} - \frac{1}{u} = 0,$$

which expresses the fact that M bounds equivariantly.

Another classical application of the localisation formula is the Atiyah–Hirzebruch formula expressing the χ_y-genus of a complex S^1-manifold in terms of the fixed point data.

We discuss a generalisation of this formula due to Krichever. It refers to the $\chi_{a,b}$-genus corresponding to the series

$$f(x) = \frac{e^{ax} - e^{bx}}{ae^{bx} - be^{ax}} \in \mathbb{Q}[a, b].$$

Particular cases are

- the top Chern number $c_n[M]$ $(a = b = -1)$;
- the signature $L[M] = \mathrm{sign}(M)$ $(a = 1,\ b = -1)$;
- Todd genus $\mathrm{td}(M)$ $(a = 0,\ b = -1)$.

Given a T^k-manifold M, choose a circle subgroup in T^k defined by a primitive vector $\nu \in \mathbb{Z}^k$:

$$S(\nu) = \{(e^{2\pi i \nu_1 \varphi}, \dots, e^{2\pi i \nu_k \varphi}) \in T^k : \varphi \in \mathbb{R}\}.$$

We have $M^{S(\nu)} = M^T$ for a generic circle $S(\nu)$. The weights of the tangential representation of $S(\nu)$ at p are $\langle \boldsymbol{w}_j(p), \nu \rangle$, $1 \leqslant j \leqslant n$.

If the fixed points M^T are isolated and $M^{S(\nu)} = M^T$, then

$$\langle \boldsymbol{w}_j(p), \nu \rangle \neq 0 \qquad \text{for } 1 \leqslant j \leqslant n \text{ and any } p \in M^T.$$

Define the index $\operatorname{ind}_\nu p$ as the number of negative weights at p, i. e.

$$\operatorname{ind}_\nu p = \#\{j : \langle \boldsymbol{w}_j(p), \nu \rangle < 0\}.$$

Theorem (generalised Atiyah–Hirzebruch formula)

The $\chi_{a,b}$-genus is T^k-rigid.
Furthermore, the $\chi_{a,b}$-genus of a stably tangentially complex $2n$-dimensional T^k-manifold M with finite M^T is given by

$$\chi_{a,b}(M) = \sum_{p \in M^T} \sigma(p)(-a)^{\operatorname{ind}_\nu p}(-b)^{n-\operatorname{ind}_\nu p}$$

for any $\nu \subset \mathbb{Z}^k$ satisfying $M^{S(\nu)} = M^T$.

Proof. The localisation formula gives

$$\chi^{S^1}_{a,b}(M) = \sum_{p \in M^T} \sigma(p) \prod_{j=1}^{n} \frac{ae^{b\langle \boldsymbol{w}_j(p),\nu\rangle x} - be^{a\langle \boldsymbol{w}_j(p),\nu\rangle x}}{e^{a\langle \boldsymbol{w}_j(p),\nu\rangle x} - e^{b\langle \boldsymbol{w}_j(p),\nu\rangle x}}.$$

This expression belongs to $\mathbb{Z}[a,b][[x]]$ (that is, it is non-singular at zero) and its constant term is $\chi_{a,b}(M)$.

Proof (continued). Denote $\omega_j = \langle \boldsymbol{w}_j(p), \nu \rangle$ and $e^{(a-b)x} = q$; then previous the expression becomes

$$\chi^{S^1}_{a,b}(M) = \sum_{p \in M^T} \sigma(p) \prod_{j=1}^{n} \frac{a - b\, q^{\omega_j}}{q^{\omega_j} - 1}.$$

Now we let $q \to \infty$. Then each factor above has limit $-b$ if $\omega_j > 0$ and limit $-a$ if $\omega_j < 0$. Therefore,

$$\lim_{q \to \infty} \chi^{S^1}_{a,b}(M) = \sum_{p \in M^T} \sigma(p)(-a)^{\mathrm{ind}_\nu\, p}(-b)^{n - \mathrm{ind}_\nu\, p}.$$

Similarly, letting $q \to 0$ we get

$$\lim_{q \to 0} \chi^{S^1}_{a,b}(M) = \sum_{p \in M^T} \sigma(p)(-a)^{n - \mathrm{ind}_\nu\, p}(-b)^{\mathrm{ind}_\nu\, p}.$$

To finish the proof, one needs to show that $\chi^{S^1}_{a,b}(M)$ is constant in q; then it coincides with either of the limits above. This is done by showing that $\chi^{S^1}_{a,b}(M)$, as a meromorphic function in $q \in \mathbb{C}$, does not have poles. $\square$

Quasitoric manifolds (after Davis and Januszkiewicz)

$2n$-dimensional manifolds M with an action of the torus T^n satisfying

- the T^n-action is locally standard (locally looks like the standard T^n-representation in $\mathbb{C}^n$);
- there is a projection $\pi\colon M \to P$ onto a simple n-polytope P whose fibres are T^n-orbits.

Examples include projective smooth toric varieties and compact symplectic $2n$-manifolds M with Hamiltonian actions of T^n

In their turn, quasitoric manifolds are examples of torus manifolds of Hattori–Masuda.

Quasitoric manifolds M provide a vast source of examples of stably complex T^n-manifolds with isolated fixed points, for which calculations with the fixed point data and Hirzebruch genera can be made explicit.

Every quasitoric M is determined by the characteristic pair (P, Λ), where P is a simple n-polytope with m facets $F_1, \dots, F_m$,
Λ is an integral $n \times m$ matrix.

The columns λ_i of Λ determine circle subgroups in T^n fixing pointwise the facial (characteristic) submanifolds $\pi^{-1}(F_i) \subset M$, $i = 1, \dots, m$.

Given a fixed point $p = F_{j_1} \cap \dots \cap F_{j_n}$ denote
$\boldsymbol{a}_{j_1}, \dots, \boldsymbol{a}_{j_n}$ the inward-pointing normals to the facets $F_{j_1}, \dots, F_{j_n}$,
$\boldsymbol{w}_{j_1}(p), \dots, \boldsymbol{w}_{j_n}(p)$ the conjugate basis to $\lambda_{j_1}, \dots, \lambda_{j_n}$.

Proposition

- $\sigma(p) = \operatorname{sign}(\det(\boldsymbol{a}_{j_1}, \dots, \boldsymbol{a}_{j_n}) \det(\lambda_{j_1}, \dots, \lambda_{j_n}))$
- *the weight vectors at p are $\boldsymbol{w}_{j_1}(p), \dots, \boldsymbol{w}_{j_n}(p)$.*

The following result can be proved by application of the localisation formula for quasitoric manifolds:

Theorem (Musin)

The 2-parameter genus $\chi_{a,b}$ is universal for T^k-rigid genera.
In particular, any T^k-rigid rational genus is $\chi_{a,b}$ for some rational parameters a, b.

Proof. We have already seen that $\chi_{a,b}$ is rigid. To see that any T^k-rigid genus is $\chi_{a,b}$ we solve the functional equation arising from the localisation formula for one particular example of T^k-manifold; the general solution will produce the required form of the series $f(x)$.

Proof (continued). We consider the quasitoric manifold $M = \mathbb{C}P^2$ with a nonstandard stably complex structure corresponding to the characteristic matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

It has three fixed points v_1, v_2, v_3 with
the signs $\sigma(v_1) = -1$, $\sigma(v_2) = 1$, $\sigma(v_3) = 1$
and the weights $\{(1,0),(1,1)\}$, $\{(0,-1),(1,1)\}$, $\{(0,1),(1,0)\}$.

Plugging these data into the localisation formula we obtain that a genus φ is rigid on M iff its defining series $f(x)$ satisfies the equation

$$-\frac{1}{f(x_1)f(x_1+x_2)} + \frac{1}{f(-x_2)f(x_1+x_2)} + \frac{1}{f(x_1)f(x_2)} = c.$$

Interchanging x_1 and x_2 gives

$$-\frac{1}{f(x_2)f(x_1+x_2)} + \frac{1}{f(-x_1)f(x_1+x_2)} + \frac{1}{f(x_2)f(x_1)} = c\,,$$

Proof (continued). Subtraction yields

$$\left(\frac{1}{f(x_1)}+\frac{1}{f(-x_1)}\right)\frac{1}{f(x_1+x_2)} \;=\; \left(\frac{1}{f(x_2)}+\frac{1}{f(-x_2)}\right)\frac{1}{f(x_1+x_2)}\,.$$

It follows that

$$\frac{1}{f(x)}+\frac{1}{f(-x)} \;=\; c' \quad \text{and} \quad \frac{1}{f(-x)} \;=\; c'-\frac{1}{f(x)}$$

for some constant c'. Substituting in the original equation gives

$$\left(\frac{1}{f(x_1)}+\frac{1}{f(x_2)}-c'\right)\frac{1}{f(x_1+x_2)} \;=\; \frac{1}{f(x_1)f(x_2)}-c\,,$$

which rearranges to

$$f(x_1+x_2) \;=\; \frac{f(x_1)+f(x_2)-c'f(x_1)f(x_2)}{1-cf(x_1)f(x_2)}\,.$$

So f is the exponential series of the formal group law $F_{a,b}(u,v)$ corresponding to $\chi_{a,b}$, with $c'=-a-b$ and $c=ab$. □

Given a stably tangentially complex T^k-manifold (M, c_T) with the fixed point data $\{\boldsymbol{w}_j(p),\ \sigma(p)\colon p \in M^T,\ 1 \leqslant j \leqslant n\}$, we refer to

$$\sum_{p \in M^T} \sigma(p) \prod_{j=1}^{n} \frac{1}{f\big(\langle \boldsymbol{w}_j(p), \boldsymbol{x} \rangle\big)} = c$$

as the rigidity equation corresponding to M.

Its solutions $f(x)$ provide Hirzebruch genera which are rigid (or fibre multiplicative) on the particular M.

For example, the rigidity equation for the nonstandard $\mathbb{C}P^2$ described above is

$$-\frac{1}{f(x_1)f(x_1+x_2)}+\frac{1}{f(-x_2)f(x_1+x_2)}+\frac{1}{f(x_1)f(x_2)}=c.$$

Its general solution is given by $f(x)=\frac{e^{ax}-e^{bx}}{ae^{bx}-be^{ax}}$, corresponding to the $\chi_{a,b}$-genus.

On the other hand, the rigidity equation for the standard $\mathbb{C}P^2$ is

$$\frac{1}{f(x_1)f(x_1+x_2)}+\frac{1}{f(-x_1-x_2)f(-x_2)}+\frac{1}{f(-x_1)f(x_2)}=c.$$

Alongside with $f(x)=\frac{e^{ax}-e^{bx}}{ae^{bx}-be^{ax}}$, it has other analytic solutions, described by Buchstaber and Bunkova.

Homogeneous spaces of compact Lie groups provide another source of concrete examples of manifolds with torus actions and isolated fixed points. They often admit invariant almost complex structures, including integrable ones, which can be classified by the methods of representation theory. Particular examples of great importance for topology include complex and quaternionic projective spaces, the Cayley plane, Grassmann and flag manifolds.

Example

$S^6 = G_2/SU(3)$ admits a G_2-invariant complex structure.

The action of the maximal torus T^2 has 2 fixed points with the corresponding weights given by $(\boldsymbol{e}_1, \boldsymbol{e}_2, -\boldsymbol{e}_1 - \boldsymbol{e}_2)$ and $(-\boldsymbol{e}_1, -\boldsymbol{e}_2, \boldsymbol{e}_1 + \boldsymbol{e}_2)$. The resulting rigidity equation is therefore

$$\frac{1}{f(x_1)f(x_2)f(-x_1-x_2)} + \frac{1}{f(-x_1)f(-x_2)f(x_1+x_2)} = c.$$

The Krichever genus

Theorem

When $c \neq 0$, the general solution of the rigidity equation

$$\frac{1}{f(x_1)f(x_2)f(-x_1-x_2)} + \frac{1}{f(-x_1)f(-x_2)f(x_1+x_2)} = c$$

is given by $f(x) = \frac{e^{\alpha x}}{\Phi(x,z)}$, where $\Phi(x,z) = \frac{\sigma(z-x)}{\sigma(z)\sigma(x)} e^{\zeta(z)\,x}$ is the Baker–Akhiezer function of the elliptic curve $y^2 = 4x^3 - g_2 x - g_3$, $\sigma(z)$ is the Weierstrass sigma-function, $\zeta(z) = (\ln \sigma(z))'$, $\wp(z) = -(\ln \sigma(z))''$.

The Krichever genus is the Hirzebruch genus corresponding $f(x) = \frac{e^{\alpha x}}{\Phi(x,z)}$.

The universal Krichever genus φ_{K} depends on 4 parameters $\alpha, \wp(z), \wp'(z), g_2$, viewed as formal variables.

The universal elliptic φ_{ell} genus is obtained by setting $\alpha = \wp'(z) = 0$ in φ_{K}. Another particular case is $\chi_{a,b}$.

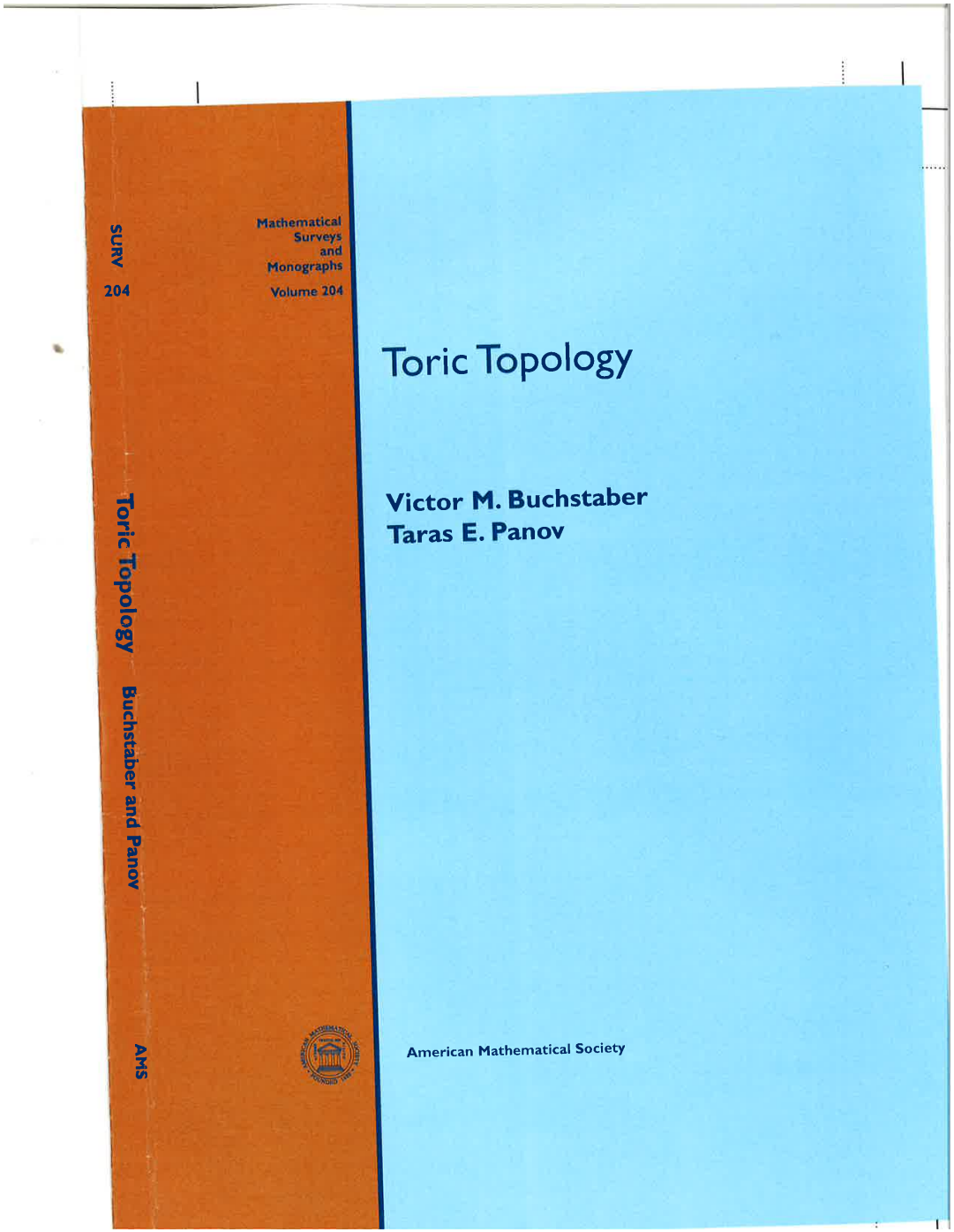
Rigidity on SU-manifolds

An SU-manifold is a stably complex manifold M with $c_1(M) = 0$.
An example is given by S^6 with the almost complex structure as above.

Krichever proved that the genus φ_{K} is rigid on SU-manifolds.
(In the oriented category, the rigidity of the elliptic genus on spin manifolds is due to Bott and Taubes.)
We therefore obtain

Theorem (Buchstaber-P-Ray)

Assume that a genus φ is rigid on SU-manifolds and $\varphi(S^6) \neq 0$. Then $\varphi = \varphi_{\mathrm{K}}$ is the Krichever genus.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