Lecture 3. Equivariant Bordism: Geometric and Homotopic Approach

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Pontryagin–Thom construction revisited

Construction (cobordism classes of η-submanifolds in $E\xi$)

ξ an m-plane bundle with total space $E\xi$ over a manifold X,
η be a k-plane bundle over a manifold Y.

An η-submanifold of $E\xi$ is a pair (M, f) consisting of a submanifold $M \subset E\xi$ and a map

$$f\colon \nu(M \subset E\xi) \to \eta$$

s. t. f is an isomorphism on each fibre, so that $\mathrm{codim}(M \subset E\xi) = k$.

Two η-submanifolds (M_0, f_0) and (M_1, f_1) are cobordant if there is an η-submanifold with boundary (W, f) in $E\xi \times I \subset E(\xi \oplus \mathbb{R})$ such that

$$\partial(W, f) = \big((M_0, f_0) \times 0\big) \cup \big((M_1, f_1) \times 1\big).$$

Given $M \subset E\xi$, get the Pontryagin–Thom collapse map

$$Th\,\xi \to Th\,\nu.$$

Theorem

There is a one-to-one correspondence

$$\{\textit{cobordism classes of } \eta\textit{-submanifolds in } E\xi\} \leftrightarrow [Th\,\xi, Th\,\eta].$$

Proof.

Assume given $g\colon\ Th\,\xi \to Th\,\eta$. By changing g within its homotopy class we may achieve that g is transverse along the zero section $Y \subset Th\,\eta$.

Since η is an n-plane bundle, $M = g^{-1}(Y)$ is a submanifold of codimension n in $E\xi = Th\,\xi \setminus pt$ with

$$\nu(M \subset E\xi) = g^*(\nu(Y \subset E\eta)) = g^*\eta.$$

That is, M is an η-submanifold in $E\xi$.

Conversely, assume given an η-submanifold $M \subset E\xi$.
Get a map of Thom spaces $Th\,\nu \to Th\,\eta$ and the Pontryagin–Thom map $Th\,\xi \to Th\,\nu$, whose composition gives the required map $Th\,\xi \to Th\,\eta$. □

Example (Unoriented bordism and cobordism)

1. $E\xi = \mathbb{R}^{k+n}$, a trivial $(k+n)$-bundle over $X = pt$ with $Th\,\xi = S^{k+n}$,
$\eta_k\colon EO(k) \to BO(k)$ the universal k-plane bundle, $Th\,\eta_k = MO(k)$,
$\eta = Y \times \eta_k$ a k-plane bundle over $Y \times BO(k)$ with $Th\,\eta = (Y_+) \wedge MO(k)$.

η-submanifolds in $E\xi$ are pairs (f, ι) with $f\colon M \to Y$ and $\iota\colon M^n \subset \mathbb{R}^{k+n}$.
By stabilisation over k we get the unoriented bordism groups

$$O_n(Y) \cong \lim_{k\to\infty} [Th\,\xi, Th(Y \times \eta_k)] = \lim_{k\to\infty} \pi_{k+n}\big((Y_+) \wedge MO(k)\big).$$

2. $E\xi = X \times \mathbb{R}^{k-n}$, a trivial $(k-n)$-bundle over X with $Th\,\xi = \Sigma^{k-n}(X_+)$,
$\eta = \eta_k\colon EO(k) \to BO(k)$ with $Th\,\eta = MO(k)$.

η-submanifolds in $E\xi$ are codimension-k embeddings $M \subset X \times \mathbb{R}^{k-n}$, or maps $M \to X$ decomposed as $M \hookrightarrow X \times \mathbb{R}^{k-n} \to X$.
By stabilisation over k we get the unoriented cobordism groups

$$O^n(X) \cong \lim_{k\to\infty} [Th(X \times \mathbb{R}^{k-n}), Th\,\eta_k] = \lim_{k\to\infty} [\Sigma^{k-n}(X_+), MO(k)].$$

Example (Complex bordism and cobordism)

1. $E\xi = \mathbb{R}^{2k+n}, \quad \eta = Y \times \eta_k \colon Y \times EU(k) \to Y \times BU(k)$.

η-submanifolds in $E\xi$ are pairs (f, ι) with $f \colon M \to Y$ and $\iota \colon M^n \subset \mathbb{R}^{2k+n}$ an embedding with a complex structure in the normal bundle.
Equivalently, one can think of a map $f \colon M \to Y$ of a tangentially stably complex manifold M to Y. By stabilisation over k we get

$$U_n(Y) \cong \lim_{k\to\infty} [Th\,\xi, Th(Y \times \eta_k)] = \lim_{k\to\infty} \pi_{2k+n}\big((Y_+) \wedge MU(k)\big).$$

2. $E\xi = X \times \mathbb{R}^{2k-n}, \quad \eta = \eta_k \colon EU(k) \to BU(k)$.

η-submanifolds in $E\xi$ are codimension-$2k$ embeddings $M \subset X \times \mathbb{R}^{2k-n}$ with a complex structure in the normal bundle, or maps $M \to X$ decomposed as $M \hookrightarrow X \times \mathbb{R}^{2k-n} \to X$.
By stabilisation over k we get the complex cobordism groups

$$U^n(X) \cong \lim_{k\to\infty} [Th(X \times \mathbb{R}^{k-n}), Th\,\eta_k] = \lim_{k\to\infty} [\Sigma^{k-n}(X_+), MO(k)].$$

A map $M \to X$ of manifolds is complex oriented if it is decomposed as

$$M \hookrightarrow X \times \mathbb{R}^{2k-n} \longrightarrow X,$$

where the first map is an embedding with a fixed structure of a complex k-plane bundle in the normal bundle.

$$U_n(X) = \{\text{bordism classes of maps } M \to X, \\ \text{where } M \text{ is tangentially stably complex of dimension } n\},$$

$$U^n(X) = \{\text{cobordism classes of complex oriented maps } M \to X \\ \text{of codimension } n\},$$

Given $y \in U^n(Y)$ represented by a complex oriented map $M \to Y$ and a transverse map $f\colon X \to Y$, the class $f^*(y) \in U^n(X)$ is represented by the pullback $X \times_Y M \to X$ with the induced complex orientation.

Pairing and products

The map $BU(k) \times BU(l) \to BU(k+l)$ classifying the product bundle induces the map of Thom spaces $MU(k) \wedge MU(l) \to MU(k+l)$.

There is a canonical pairing (the Kronecker product)

$$\langle\ ,\ \rangle\colon U^m(X) \otimes U_n(X) \to \Omega^U_{n-m},$$

the $\frown$-product

$$\frown\colon U^m(X) \otimes U_n(X) \to U_{n-m}(X),$$

and the $\smile$-product (or simply product)

$$\smile\colon U^m(X) \otimes U^n(X) \to U^{m+n}(X).$$

Their homotopical and geometric definition is given next.

Assume given $x \in U^m(X)$ represented by a map $\Sigma^{2l-m}X_+ \to MU(l)$ and $\alpha \in U_n(X)$ represented by a map $S^{2k+n} \to X_+ \wedge MU(k)$.
Then $\langle x, \alpha \rangle \in \Omega^U_{n-m}$ is represented by the composite

$$S^{2k+2l+n-m} \xrightarrow{\Sigma^{2l-m}\alpha} \Sigma^{2l-m}X_+ \wedge MU(k) \xrightarrow{x \wedge \mathrm{id}} MU(l) \wedge MU(k) \to MU(l+k)$$

If $\Delta\colon X_+ \to (X \times X)_+ = X_+ \wedge X_+$ is the diagonal map, then $x \frown \alpha \in U_{n-m}(X)$ is represented by the composite map

$$S^{2k+2l+n-m} \xrightarrow{\Sigma^{2l-m}\alpha} \Sigma^{2l-m}X_+ \wedge MU(k) \xrightarrow{\Sigma^{2l-m}\Delta \wedge \mathrm{id}} X_+ \wedge \Sigma^{2l-m}X_+ \wedge MU(k)$$
$$\xrightarrow{\mathrm{id} \wedge x \wedge \mathrm{id}} X_+ \wedge MU(l) \wedge MU(k) \to X_+ \wedge MU(l+k)$$

The $\smile$-product is defined similarly; it turns $U^*(X) = \prod_{n \in \mathbb{Z}} U^n(X)$ into a graded ring, the complex cobordism ring of X.

In geometric terms, assume $x \in U^m(X)$ is represented by an embedding $M^{k-m} \hookrightarrow X = X^k$ with a complex structure in the normal bundle, and $\alpha \in U_n(X)$ is represented by an embedding $N^n \hookrightarrow X^k$ of a tangentially stably complex manifold N. Assume further that M and N intersect transversely in X, i. e. $\dim M \cap N = n - m$. Then $\langle x, \alpha \rangle$ is the bordism class of the intersection $M \cap N$, and $x \frown \alpha$ is the bordism class of the embedding $M \cap N \to X$ with the induced tangential complex structure.

Similarly, if $x \in U^{-d}(X)$ is represented by a smooth fibre bundle $E^{k+d} \to X^k$ and $\alpha \in U_n(X)$ is represented by a smooth map $N \to X$, then $\langle x, \alpha \rangle \in \Omega^U_{n+d}$ is the bordism class of the pull-back E', and $x \frown \alpha \in U_{n+d}(X)$ is the bordism class of the composite map $E' \to X$ in the pull-back diagram

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ N & \longrightarrow & X \end{array}$$

Construction (Poincaré–Atiyah duality in complex bordism)

X a manifold of dimension d. The inclusion $pt \subset X$ defines the bordism class $1 \in U_0(X)$ and a cobordism class in $U^d(X)$. (The normal bundle of $pt \subset X$ or $pt \subset X \times \mathbb{R}$ has a complex structure.)

The identity $X \to X$ defines the cobordism class $1 \in U^0(X)$. It defines the fundamental bordism class of X in $U_d(X)$ only when X is stably complex.

Now let X be stably complex manifold with fundamental bordism class $[X] \in U_d(X)$. There is an isomorphism

$$D = \cdot \frown [X] \colon U^k(X) \to U_{d-k}(X), \quad x \mapsto x \frown [X],$$

called the Poincaré–Atiyah duality map.

Construction (Gysin homomorphism)

Let $f\colon X^k \to Y^{k+d}$ be a complex oriented map of codimension d between manifolds (manifolds may not be compact, in which case f is assumed to be proper). It induces a covariant map

$$f_!\colon U^n(X) \to U^{n+d}(Y)$$

called the Gysin homomorphism, whose geometric definition is as follows. Let $x \in U^n(X)$ be represented by a complex oriented map $g\colon M^{k-n} \to X^k$. Then $f_!(x)$ is represented by the composition fg.

Proposition

The Gysin homomorphism $f_!\colon U^*(X) \to U^{*+d}(Y)$ is a Ω_U-module map depending only on the proper homotopy class of f. Furthermore, it satisfies

(a) $f_!(x \cdot f^*(y)) = f_!(x) \cdot y$ *for any* $x \in U^n(X)$, $y \in U^m(Y)$;

(b) *given a pullback square*

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y \end{array}$$

with g transverse to f, one has $g^* f_! = f'_! g'^*\colon U^*(X) \to U^{*+d}(Z)$.

Proof.

(a) Choose maps $Z \to X$ and $W \to Y$ representing x and y, and consider

$$\begin{array}{ccccc} Z \times_Y W & \longrightarrow & X \times_Y W & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longrightarrow & X & \longrightarrow & Y \end{array}$$

Both sides of (a) are represented by the composite $Z \times_Y W \to Y$. □

We now proceed to describe the theory of complex bordism and genera in the equivariant setting.

We consider stably complex manifolds equipped with compatible actions of a torus T^k. (With little effort the theory can be extended to actions of compact connected Lie groups.)

The theory focuses on the notion of universal toric genus Φ, defined on stably complex T^k-manifolds and taking values in $U^*(BT^k)$.

Φ is an equivariant analogue of the universal genus $\mathrm{id}\colon \Omega_U \to \Omega_U$.

The idea is to define a transformation of equivariant cobordism functors

$$\varPhi_X\colon U^*_{T^k}(X) \xrightarrow{\nu} MU^*_{T^k}(X) \xrightarrow{\alpha} U^*(ET^k \times_{T^k} X).$$

Here $U^*_{T^k}(X)$ is the geometric and $MU^*_{T^k}(X))$ the homotopic T^k-equivariant complex cobordism ring of a T^k-manifold X, and $ET^k \times_{T^k} X$ is the Borel construction.

The geometric and homotopical versions of equivariant cobordism are different, because of the lack of equivariant transversality.

By restricting $\varPhi_X$ to the case $X = pt$ we get a homomorphism

$$\varPhi\colon \varOmega_{U:T^k} \to \varOmega_U[[u_1, \ldots, u_k]]$$

from the geometric T^k-cobordism ring $\varOmega^*_{U:T^k} := U^*_{T^k}(pt)$ to the ring $U^*(BT^k) = \varOmega_U[[u_1, \ldots, u_k]]$. We refer to $\varPhi$ as the universal toric genus. It assigns to a bordism class $[M, c_T] \in \varOmega^{U:T^k}_{2n}$ of a stably complex T^k-manifold M the 'cobordism class' of the map $ET^k \times_{T^k} M \to BT^k$.

Homotopic equivariant cobordism (after tom Dieck, Löffler)

Thom T^k-spectrum MU_{T^k} has spaces indexed by the inclusion poset of complex representations V of T^k (of complex dimension $|V|$).

Each $MU_{T^k}(V)$ is the Thom T^k-space of the universal $|V|$-dimensional complex T^k-equivariant vector bundle $\gamma_V \colon EU_{T^k}(V) \to BU_{T^k}(V)$, and each spectrum map $\Sigma^{2(|W|-|V|)} MU_{T^k}(V) \to MU_{T^k}(W)$ is induced by the inclusion $V \subset W$ of a T^k-submodule.

The homotopic T^k-equivariant complex cobordism group $MU^n_{T^k}(X)$ of a pointed T^k-space X is defined by stabilising the pointed T^k-homotopy sets:

$$MU^n_{T^k}(X) = \varinjlim \big[\Sigma^{2|V|-n}(X_+), MU_{T^k}(V)\big]_{T^k}.$$

Applying the Borel construction to $\gamma_V \colon EU_{T^k}(V) \to BU_{T^k}(V)$ yields a complex $|V|$-dimensional bundle $ET^k \times_{T^k} \gamma_V$ over $ET^k \times_{T^k} BU_{T^k}(V)$, whose Thom space is $ET^k_+ \wedge_{T^k} MU_{T^k}(V)$.
The classifying map for $ET^k \times_{T^k} \gamma_V$ induces a map of Thom spaces $ET^k_+ \wedge_{T^k} MU_{T^k}(V) \to MU(|V|)$.

Now consider a T^k-map $\varSigma^{2|V|-n}(X_+) \to MU_{T^k}(V)$ representing a cobordism class in $MU^n_{T^k}(X)$. By applying the Borel construction get

$$\varSigma^{2|V|-n}(ET^k \times_{T^k} X)_+ \longrightarrow ET^k_+ \wedge_{T^k} MU_{T^k}(V) \to MU(|V|).$$

This construction is homotopy invariant, so we get a map

$$\left[\varSigma^{2|V|-n}(X_+), MU_{T^k}(V)\right]_{T^k} \longrightarrow \left[\varSigma^{2|V|-n}(ET^k \times_{T^k} X)_+, MU(|V|)\right].$$

and a multiplicative transformation of cohomology theoires

$$\alpha\colon MU^*_{T^k}(X) \longrightarrow U^*(ET^k \times_{T^k} X).$$

The construction of α may be also interpreted using the homomorphism

$$MU^*_{T^k}(X) \to MU^*_{T^k}(ET^k \times X)$$

induced by the T^k-projection $ET^k \times X \to X$; since T^k acts freely on $ET^k \times X$, the target may be replaced by $U^*(ET^k \times_{T^k} X)$.

Moreover, $\alpha\colon MU^*_{T^k}(X) \longrightarrow U^*(ET^k \times_{T^k} X)$ is an isomorphism whenever X is compact and T^k acts freely.

Remark

According to a result of Löffler, α is the homomorphism of completion with respect to the augmentation ideal in $MU^*_{T^k}(X)$.

Geometric equivariant cobordism

There is an equivariant version of Quillen's geometric approach to complex cobordism via complex oriented maps.

However, this approach relies on normal complex structures, whereas most of our examples are presented in terms of tangential information. In the equivariant situation, tangential structures may be converted to normal, but the procedure is not reversible.

For a manifold X, elements of $U^{-d}(X)$ can be represented by stably tangentially complex bundles $\pi\colon E \to X$ with d-dimensional fibre F, i. e. by those π for which the bundle $\mathcal{T}_F(E)$ of tangents along the fibre is equipped with a stably complex structure $c_{\mathcal{T}}(\pi)$.

If $\pi\colon E \to X$ is T^k-equivariant bundle, then it is stably tangentially complex as a T^k-equivariant bundle when $c_{\mathcal{T}}(\pi)$ is also T^k-equivariant.

The geometric T^k-equivariant complex cobordism group $U_{T^k}^{-d}(X)$ consists of equivariant cobordism classes of d-dimensional stably tangentially complex T^k-equivariant bundles over X.

If $X = pt$, then $\Omega_{U:T^k}^{-d} := U_{T^k}^{-d}(pt)$ consists of cobordism classes of T^k-equivariant stably tangentially complex d-manifolds M. The direct sum $\Omega_{U:T^k} = \bigoplus_d \Omega_{U:T^k}^{-d}$ is the geometric T^k-equivariant cobordism ring, and $U_{T^k}^*(X)$ is a graded $\Omega_{U:T^k}$-module. Furthermore, $U_{T^k}^*(\,\cdot\,)$ is functorial with respect to pullback along smooth T^k-maps $Y \to X$.

Given a T^k-manifold X, there are canonical homomorphisms

$$\nu\colon U_{T^k}^{-d}(X) \to MU_{T^k}^{-d}(X), \quad d \geqslant 0.$$

The idea is to convert the tangential structure used in the definition of the geometric cobordism group to the normal structure required for the Pontryagin–Thom collapse map in the homotopic approach.

The universal toric genus

For any smooth compact T^k-manifold X, we define the homomorphism

$$\Phi_X\colon U^*_{T^k}(X) \xrightarrow{\nu} MU^*_{T^k}(X) \xrightarrow{\alpha} U^*(ET^k \times_{T^k} X).$$

The homomorphism

$$\Phi\colon \Omega_{U:T^k} \longrightarrow U^*(BT^k) = \Omega_U[[u_1, \ldots, u_k]]$$

corresponding to the case $X = pt$ above is called the universal toric genus.

The genus Φ is a multiplicative cobordism invariant of stably complex T^k-manifolds. As such it is an equivariant extension of Hirzebruch's original notion of genus, and is related to the theory of formal group laws. We explore this relation and study other equivariant genera in the next lectures.

By results of Hanke and Löffler, when $X = pt$, both ν and α are monomorphisms; therefore so is Φ.

In geometric terms, the universal toric genus Φ assigns to a geometric cobordism class $[M, c_{\mathcal{T}}] \in \Omega^{-d}_{U:T^k}$ of a d-dimensional stably complex T^k-manifold M the 'cobordism class' of the map $ET^k \times_{T^k} M \to BT^k$. Since both $ET^k \times_{T^k} M$ and BT^k are infinite-dimensional, one needs to use their finite approximations to define the cobordism class $\Phi(M)$ purely in terms of stably complex structures.
Here is a conceptual way to make this precise.

Proposition

Let $[M] \in \Omega_{U:T^k}$ be a geometric equivariant cobordism class represented by a d-dimensional T^k-manifold M. Then

$$\Phi(M) = (\mathrm{id} \times_{T^k} \pi)_! 1,$$

where

$$(\mathrm{id} \times_{T^k} \pi)_! \colon U^*(ET^k \times_{T^k} M) \longrightarrow U^{*-d}(ET^k \times_{T^k} pt) = U^{*-d}(BT^k)$$

is the Gysin homomorphism induced by the projection $\pi \colon M \to pt$.

Literature

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