Lecture 2. Formal Group Laws and Hirzebruch Genera

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Elements of the theory of formal group laws

R a commutative ring with unit.
A formal power series $F(u,v) \in R[[u,v]]$ is called a (commutative one-dimensional) formal group law over R if it satisfies

(a) $F(u,0) = u,\ F(0,v) = v$;
(b) $F(F(u,v),w) = F(u,F(v,w))$;
(c) $F(u,v) = F(v,u)$.

The original example of a formal group law over a field $\mathbf{k}$ is provided by the expansion near the unit of the multiplication map $G \times G \to G$ in a one-dimensional algebraic group over $\mathbf{k}$. This also explains the terminology.

A formal group law F over R is linearisable if there exists a coordinate change $u \mapsto g_F(u) = u + \sum_{i \geqslant 1} g_i u^{i+1} \in R[[u]]$ such that

$$g_F(F(u,v)) = g_F(u) + g_F(v).$$

Theorem

Every formal group law F is linearisable over $R \otimes \mathbb{Q}$.

Proof.

Consider the series $\omega(u) = \frac{\partial F(u,w)}{\partial w}\big|_{w=0}$. Applying $\frac{\partial}{\partial w}\big|_{w=0}$ to both sides of the identity $F(F(u,v),w) = F(u,F(v,w))$ we obtain

$$\omega(F(u,v)) = \frac{\partial F(F(u,v),w)}{\partial w}\Big|_{w=0} = \frac{\partial F(u,F(v,w))}{\partial F(v,w)} \cdot \frac{\partial F(v,w)}{\partial w}\Big|_{w=0} = \frac{\partial F(u,v)}{\partial v}\omega(v).$$

We therefore have $\frac{dv}{\omega(v)} = \frac{dF(u,v)}{\omega(F(u,v))}$, where u is a parameter. Set

$$g(u) = \int_0^u \frac{dv}{\omega(v)}.$$

Integrating the identity $\frac{dv}{\omega(v)} = \frac{dF(u,v)}{\omega(F(u,v))}$ we obtain

$$g(w) = \int_0^w \frac{dv}{\omega(v)} = \int_0^w \frac{dF(u,v)}{\omega(F(u,v))} = \int_u^{F(u,w)} \frac{dt}{\omega(t)} = g(F(u,w)) - g(u),$$

so that g is a linearisation of F. □

A series $g_F(u) = u + \sum_{i\geqslant 1} g_i u^{i+1}$ satisfying $g_F(F(u,v)) = g_F(u) + g_F(v)$ is called the logarithm of the formal group law F. Its functional inverse series $f_F(t) \in R \otimes \mathbb{Q}[[t]]$ is the exponential of F, so we have $F(u,v) = f_F(g_F(u) + g_F(v))$ over $R \otimes \mathbb{Q}$.

If R does not have torsion (i.e. $R \to R \otimes \mathbb{Q}$ is monomorphic), then a formal group law is fully determined by its logarithm.

Example

The multiplicative formal group law is the series

$$F(u,v) = (1+u)(1+v) - 1 = u + v + uv.$$

There is a 1-parameter graded extension given by

$$F_\beta(u,v) = u + v - \beta uv, \quad \deg \beta = -2,$$

with coefficients in $\mathbb{Z}[\beta]$. Its exponential and logarithm are given by

$$f(x) = \frac{1 - e^{-\beta x}}{\beta}, \quad g(u) = -\frac{\ln(1-\beta u)}{\beta} \in \mathbb{Q}[\beta].$$

Example

Another classical example comes from the theory of elliptic functions. There is a unique meromorphic function $f(x)$ with $f(0)=0$ and $f'(0)=1$ satisfying the differential equation

$$(f'(x))^2 = 1 - 2\delta f^2(x) + \varepsilon f^4(x)$$

with $\delta, \varepsilon \in \mathbb{C}$. This function provides a uniformisation for the Jacobi model $y^2 = 1 - 2\delta x^2 + \varepsilon x^4$ of an elliptic curve. When the discriminant

$$\Delta = \varepsilon(\delta^2 - \varepsilon)$$

is nonzero, the elliptic curve is nondegenerate, and $f(x)$ is a doubly periodic function known as the Jacobi elliptic sine and denoted by $\mathrm{sn}(x)$. Its inverse is given by the elliptic integral

$$g(u) = \int_0^u \frac{dt}{\sqrt{1 - 2\delta t^2 + \varepsilon t^4}}.$$

Example

There is the following Euler's expression for the addition formula for $\mathrm{sn}(x)$:

$$F_{\mathrm{ell}}(u,v)=\mathrm{sn}(x+y)=\frac{u\sqrt{1-2\delta v^2+\varepsilon v^4}+v\sqrt{1-2\delta u^2+\varepsilon u^4}}{1-\varepsilon u^2v^2},$$

where $u=\mathrm{sn}\,x$, $v=\mathrm{sn}\,y$. It defines the elliptic formal group law, with exponential $\mathrm{sn}(x)$ and logarithm $g(u)$ as above.

Viewing δ, ε as formal parameters with $\deg\delta=-4$, $\deg\varepsilon=-8$, we obtain the universal elliptic formal group law over the ring $\mathbb{Z}[\frac{1}{2}][\delta,\varepsilon]$.

Degeneration $\varepsilon=0$ gives the addition formula for $f(x)=\frac{\sin\sqrt{2\delta}x}{\sqrt{2\delta}}$, while degeneration $\varepsilon=\delta^2$ gives the addition formula for $f(x)=\frac{\tanh\sqrt{\delta}x}{\sqrt{\delta}}$.

Given a ring homomorphism $r\colon R \to R'$ and a f.g.l. $F = \sum_{k,l} a_{kl} u^k v^l$ over R, we obtain a f.g.l. $r(F) := \sum_{k,l} r(a_{kl}) u^k v^l \in R'[[u,v]]$ over R'.

A formal group law $\mathcal{F}$ over a ring A is universal if for any f.g.l. F over any ring R there exists a unique homomorphism $r\colon A \to R$ such that $F = r(\mathcal{F})$.

Proposition

A universal formal group law $\mathcal{F}(u,v) = u + v + \sum_{k\geqslant 1, l \geqslant 1} a_{kl} u^k v^l$ exists, and its coefficient ring is the quotient

$$A = \mathbb{Z}[\, a_{kl}\colon k \geqslant 1,\ l \geqslant 1]/\mathcal{I}, \quad \deg a_{kl} = -2(k+l-1),$$

of the graded polynomial ring by the graded 'associativity ideal' $\mathcal{I}$, generated by the coefficients of the formal power series $\mathcal{F}(\mathcal{F}(u,v),w) - \mathcal{F}(u,\mathcal{F}(v,w))$.

Furthermore, $\mathcal{F}$ is unique: if $\mathcal{F}'$ is another universal formal group law over A', then there is an isomorphism $r\colon A \to A'$ such that $\mathcal{F}' = r(\mathcal{F})$.

Note that the definition of a formal group law does not assume any grading of the coefficient ring; however, the coefficient ring of the universal formal group law turns out to be naturally graded.

Natural grading: $\deg u = \deg v = 2$, $\deg a_{kl} = -2(k+l-1)$;
then the whole expression

$$\mathcal{F}(u,v) = u + v + \sum_{k\geqslant 1, l\geqslant 1} a_{kl} u^k v^l$$

is homogeneous of degree 2.

Theorem (Lazard)

The coefficient ring A of the universal formal group law $\mathcal{F}$ is isomorphic to the graded polynomial ring $\mathbb{Z}[a_1, a_2, \ldots]$ on an infinite number of generators, $\deg a_i = -2i$.

Construction (geometric cobordisms)

For any cell complex X, have $H^2(X) = [X, \mathbb{C}P^\infty]$. Since $\mathbb{C}P^\infty = MU(1)$, every element $x \in H^2(X)$ determines a cobordism class $u_x \in U^2(X)$, a geometric cobordism. Hence, $H^2(X) \subset U^2(X)$ (a subset, not a subgroup!)

When X is a manifold, each $u_x \in U^2(X)$ corresponds to a submanifold $M \subset X$ of codimension 2 with a complex structure on the normal bundle.

Indeed, $x \in H^2(X)$ corresponds to a homotopy class of $f_x\colon X \to \mathbb{C}P^\infty$. May assume $f_x(X)$ is transverse to a hyperplane $H \subset \mathbb{C}P^N \subset \mathbb{C}P^\infty$. Then $M_x = f_x^{-1}(H)$ is a codimension-2 submanifold in X.
A homotopy of f_x gives a cobordism of $M_x \to X$.

Conversely, given an embedding $i\colon M \subset X$ as above, the composite $X \to Th(\nu) \to MU(1) = \mathbb{C}P^\infty$ of the Pontryagin–Thom collapse map and the classifying map for ν defines an element $x_M \in H^2(X)$, and therefore a geometric cobordism.

If X is oriented, then $i_*\langle M\rangle \in H_*(X)$ is Poincaré dual to $x_M \in H^2(X)$.

Ring generators for Ω^U

As we have seen, the characteristic number s_n vanishes on decomposable elements of Ω^U. Furthermore, this characteristic number detects indecomposables that may be chosen as polynomial generators:

Theorem

A bordism class $[M] \in \Omega^U_{2n}$ may be chosen as a polynomial generator a_n of the ring Ω^U if and only if

$$s_n[M] = \begin{cases} \pm 1 & \text{if } n \neq p^k - 1 \text{ for any prime } p; \\ \pm p & \text{if } n = p^k - 1 \text{ for some prime } p. \end{cases}$$

There is no universal description of manifolds representing the polynomial generators $a_n \in \Omega^U$. On the other hand, there is a particularly nice family of manifolds whose bordism classes generate the whole ring Ω^U. This family is redundant though, so there are algebraic relations between their bordism classes.

Construction (Milnor hypersurfaces)

The Milnor hypersurface in $\mathbb{C}P^i \times \mathbb{C}P^j$ $(0 \leqslant i \leqslant j)$ is

$$H_{ij} = \{(z_0 : \cdots : z_i) \times (w_0 : \cdots : w_j) \in \mathbb{C}P^i \times \mathbb{C}P^j \colon z_0 w_0 + \cdots + z_i w_i = 0\}$$

Note that $H_{0j} \cong \mathbb{C}P^{j-1}$.

More intrinsically, H_{ij} is a hyperplane section of the Segre embedding

$$\sigma \colon \mathbb{C}P^i \times \mathbb{C}P^j \to \mathbb{C}P^{(i+1)(j+1)-1},$$
$$(z_0 : \cdots : z_i) \times (w_0 : \cdots : w_j) \mapsto (z_0 w_0 : z_0 w_1 : \cdots : z_k w_l : \cdots : z_i w_j),$$

Also, H_{ij} may be identified with the set of pairs (ℓ, α), where ℓ is a line in $\mathbb{C}^{i+1}$ and α is a hyperplane in $\mathbb{C}^{j+1}$ containing ℓ.
In particular, $H_{22} = Fl(\mathbb{C}^3)$, the flag manifold.
The projection $H_{ij} \to \mathbb{C}P^i$, $(\ell, \alpha) \mapsto \ell$, is a fibre bundle with fibre $\mathbb{C}P^{j-1}$.

Denote by p_1 and p_2 the projections of $\mathbb{C}P^i \times \mathbb{C}P^j$ onto its factors. Then

$$H^*(\mathbb{C}P^i \times \mathbb{C}P^j) = \mathbb{Z}[x, y]/(x^{i+1} = 0,\ y^{j+1} = 0)$$

where $x = p_1^* c_1(\bar{\eta})$, $y = p_2^* c_1(\bar{\eta})$, and η the tautological bundle.

Proposition

H_{ij} represents the geometric cobordism in $\mathbb{C}P^i \times \mathbb{C}P^j$ corresponding $x + y \in H^2(\mathbb{C}P^i \times \mathbb{C}P^j)$. In particular, the image of the fundamental class $\langle H_{ij} \rangle$ in $H_{2(i+j-1)}(\mathbb{C}P^i \times \mathbb{C}P^j)$ is Poincaré dual to $x + y$.

Proof.

We have $x + y = c_1(p_1^*(\bar{\eta}) \otimes p_2^*(\bar{\eta}))$. The classifying map for $p_1^*(\bar{\eta}) \otimes p_2^*(\bar{\eta})$ is the Segre embedding $\sigma\colon \mathbb{C}P^i \times \mathbb{C}P^j \to \mathbb{C}P^{(i+1)(j+1)-1} \to \mathbb{C}P^\infty$.
The codimension-2 submanifold in $\mathbb{C}P^i \times \mathbb{C}P^j$ corresponding to $x + y$ is the preimage $\sigma^{-1}(H)$ of a generally positioned hyperplane in $\mathbb{C}P^{(i+1)(j+1)-1}$. The Milnor hypersurface H_{ij} is exactly $\sigma^{-1}(H)$ for one such hyperplane H. □

Lemma

$$s_{i+j-1}[H_{ij}] = \begin{cases} j & \textit{if} \;\; i = 0; \\ -\binom{i+j}{i} & \textit{if} \;\; i > 1. \end{cases}$$

Proof.

The stably complex structure on $H_{0j} = \mathbb{C}P^{j-1}$ comes from the isomorphism $\mathcal{T}(\mathbb{C}P^{j-1}) \oplus \mathbb{C} \cong \bar{\eta} \oplus \cdots \oplus \bar{\eta}$ (j summands) and $x = c_1(\bar{\eta})$, so have

$$s_{j-1}[\mathbb{C}P^{j-1}] = jx^{j-1}\langle \mathbb{C}P^{j-1} \rangle = j.$$

Denote by ν the normal bundle of $\iota\colon H_{ij} \hookrightarrow \mathbb{C}P^i \times \mathbb{C}P^j$. Then

$$\mathcal{T}(H_{ij}) \oplus \nu = \iota^*(\mathcal{T}(\mathbb{C}P^i \times \mathbb{C}P^j)).$$

We have $s_{i+j-1}(\nu) = \iota^*(x+y)^{i+j-1}$ and
$s_{i+j-1}(\mathcal{T}(\mathbb{C}P^i \times \mathbb{C}P^j)) = (i+1)x^{i+j-1} + (j+1)y^{i+j-1} = 0$ for $i > 1$, so

$$\begin{aligned} s_{i+j-1}[H_{ij}] &= -s_{i+j-1}(\nu)\langle H_{ij} \rangle = -\iota^*(x+y)^{i+j-1}\langle H_{ij} \rangle \\ &= -(x+y)^{i+j}\langle \mathbb{C}P^i \times \mathbb{C}P^j \rangle = -\tbinom{i+j}{i}. \end{aligned}$$

□

Theorem

The bordism classes $\{[H_{ij}], 0 \leqslant i \leqslant j\}$ *generate the ring* Ω^U.

Proof.

$$\text{g.c.d.}\Big(\binom{n+1}{i},\ 1 \leqslant i \leqslant n\Big) = \begin{cases} p & \text{if } n = p^k - 1, \\ 1 & \text{otherwise.} \end{cases}$$

Now the previous calculation of $s_{i+j-1}[H_{ij}]$ implies that a certain integer linear combination of bordism classes $[H_{ij}]$ with $i + j = n + 1$ has s_{i+j-1} equal p or 1, as needed for the polynomial generator a_n. □

Example

- $\Omega^U_2 = \mathbb{Z}$, generated by $[\mathbb{C}P^1]$, as $1 = 2^1 - 1$ and $s_1[\mathbb{C}P^1] = 2$;
- $\Omega^U_4 = \mathbb{Z} \oplus \mathbb{Z}$, generated by $[\mathbb{C}P^1 \times \mathbb{C}P^1]$ and $[\mathbb{C}P^2]$, as $2 = 3^1 - 1$ and $s_2[\mathbb{C}P^2] = 3$;
- $[\mathbb{C}P^3]$ cannot be taken as the polynomial generator $a_3 \in \Omega^U_6$, since $s_3[\mathbb{C}P^3] = 4$, while $s_3(a_3) = \pm 2$. We have $a_3 = [H_{22}] + [\mathbb{C}P^3]$.

Formal group law of geometric cobordisms

The applications of formal group laws in cobordism theory build upon the following fundamental construction due to Novikov.

Let X be a cell complex and $u, v \in U^2(X)$ two geometric cobordisms corresponding to elements $x, y \in H^2(X)$ respectively. Denote by $u +_{H} v$ the geometric cobordism corresponding to the cohomology class $x + y$.

Proposition

The following relation holds in $U^2(X)$:

$$u +_{H} v = F_U(u, v) = u + v + \sum_{k \geqslant 1,\, l \geqslant 1} \alpha_{kl}\, u^k v^l,$$

where the coefficients $\alpha_{kl} \in \Omega_U^{-2(k+l-1)}$ do not depend on X. The series $F_U(u, v)$ is a formal group law over the complex cobordism ring Ω_U.

Proof.

We first calculate on the universal example $X = \mathbb{C}P^\infty \times \mathbb{C}P^\infty$. Then

$$U^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \Omega_U[[\underline{u}, \underline{v}]],$$

where $\underline{u}, \underline{v}$ are canonical geometric cobordisms given by the projections of $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ onto its factors.
We therefore have the following relation in $U^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$:

$$\underline{u} +_H \underline{v} = \sum_{k,l \geqslant 0} \alpha_{kl}\, \underline{u}^k \underline{v}^l,$$

where $\alpha_{kl} \in \Omega_U^{-2(k+l-1)}$.

Now let the geometric cobordisms $u, v \in U^2(X)$ be given by maps $f_u, f_v \colon X \to \mathbb{C}P^\infty$ respectively. Then $u = (f_u \times f_v)^*(\underline{u})$, $v = (f_u \times f_v)^*(\underline{v})$ and $u +_H v = (f_u \times f_v)^*(\underline{u} +_H \underline{v})$, where $f_u \times f_v \colon X \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$.
Applying the Ω_U-module map $(f_u \times f_v)^*$ to the above expression gives the required formula. □

The series $u +_H v = F_U(u,v)$ is called the formal group law of geometric cobordisms, or simply the formal group law of complex cobordism.

By definition, the geometric cobordism $u \in U^2(X)$ is the first Conner–Floyd Chern class $c_1^U(\xi)$ of the complex line bundle ξ over X obtained by pulling back the conjugate tautological bundle along the map $f_u \colon X \to \mathbb{C}P^\infty$.

It follows that the formal group law of geometric cobordisms gives an expression of $c_1^U(\xi \otimes \eta) \in U^2(X)$ in terms of the classes $u = c_1^U(\xi)$ and $v = c_1^U(\eta)$ of the factors:

$$c_1^U(\xi \otimes \eta) = F_U(u,v).$$

Theorem (Buchstaber)

$$F_U(u,v) = \frac{\sum_{i,j\geqslant 0}[H_{ij}]u^i v^j}{\left(\sum_{r\geqslant 0}[\mathbb{C}P^r]u^r\right)\left(\sum_{s\geqslant 0}[\mathbb{C}P^s]v^s\right)},$$

where H_{ij} $(0 \leqslant i \leqslant j)$ *are Milnor hypersurfaces and* $H_{ji} = H_{ij}$.

Proof.

Consider the Poincaré–Atiyah duality map
$D\colon U^2(\mathbb{C}P^i \times \mathbb{C}P^j) \to U_{2(i+j)-2}(\mathbb{C}P^i \times \mathbb{C}P^j)$ and the augmentation
$\varepsilon\colon U_*(\mathbb{C}P^i \times \mathbb{C}P^j) \to U_*(pt) = \Omega^U$.
The composite $\varepsilon D\colon U^2(\mathbb{C}P^i \times \mathbb{C}P^j) \to \Omega^U_{2(i+j)-2}$ takes geometric cobordisms to the bordism classes of the corresponding submanifolds.
In particular, $\varepsilon D(u +_H v) = [H_{ij}]$, $\varepsilon D(u^k v^l) = [\mathbb{C}P^{i-k}][\mathbb{C}P^{j-l}]$. Applying εD to $u +_H v = F_U(u,v)$ we get $[H_{ij}] = \sum_{k,\,l} \alpha_{kl}[\mathbb{C}P^{i-k}][\mathbb{C}P^{j-l}]$.Therefore,

$$\sum_{i,j}[H_{ij}]u^i v^j = \Bigl(\sum_{k,\,l}\alpha_{kl}u^k v^l\Bigr)\Bigl(\sum_{i\geqslant k}[\mathbb{C}P^{i-k}]u^{i-k}\Bigr)\Bigl(\sum_{j\geqslant l}[\mathbb{C}P^{j-l}]v^{j-l}\Bigr).$$ □

Corollary

The coefficients of the formal group law of geometric cobordisms generate the complex cobordism ring Ω_U.

Theorem (Mishchenko)

The logarithm of the formal group law of geometric cobordisms is given by

$$g_U(u) = u + \sum_{k\geqslant 1} [\mathbb{C}P^k] \frac{u^{k+1}}{k+1} \in \Omega_U \otimes \mathbb{Q}[[u]].$$

Proof.

$$\frac{dg_U(u)}{du} = \frac{1}{\left.\frac{\partial F_U(u,v)}{\partial v}\right|_{v=0}} = \frac{1 + \sum_{k>0} [\mathbb{C}P^k] u^k}{1 + \sum_{i>0} ([H_{i1}] - [\mathbb{C}P^1][\mathbb{C}P^{i-1}]) u^i}.$$

Now $[H_{i1}] = [\mathbb{C}P^1][\mathbb{C}P^{i-1}]$ (by calculating the Chern numbers), which gives $\frac{dg_U(u)}{du} = 1 + \sum_{k>0} [\mathbb{C}P^k] u^k$. □

Theorem (Quillen)

The formal group law F_U of geometric cobordisms is universal.

Proof.

Let $\mathcal{F}$ be the universal formal group law over a ring A. Then there is a homomorphism $r\colon A \to \Omega_U$ which takes $\mathcal{F}$ to F_U.

The series $\mathcal{F}$, viewed as a f.g.l. over the ring $A \otimes \mathbb{Q}$, has the universality property for all f.g.l. over $\mathbb{Q}$-algebras. Writing the logarithm of $\mathcal{F}$ as $\sum b_k \frac{u^{k+1}}{k+1}$ we obtain that $A \otimes \mathbb{Q} = \mathbb{Q}[b_1, b_2, \ldots]$.

By Mishchenko's formula for the logarithm, $r(b_k) = [\mathbb{C}P^k] \in \Omega_U$. Since $\Omega_U \otimes \mathbb{Q} \cong \mathbb{Q}[[\mathbb{C}P^1], [\mathbb{C}P^2], \ldots]$, this implies that $r \otimes \mathbb{Q}$ is an isomorphism.

By Lazard's Theorem the ring A does not have torsion, so r is a monomorphism. On the other hand, Buchstaber's formula for $F_U(u, v)$ implies that the image $r(A)$ contains the bordism classes $[H_{ij}] \in \Omega_U$, $0 \leqslant i \leqslant j$. Since these classes generate the whole ring Ω_U, the map r is onto and thus an isomorphism. □

Hirzebruch genera (complex case)

Every homomorphism $\varphi\colon \Omega^U \to R$ from the complex bordism ring to a commutative ring R with unit can be regarded as a multiplicative characteristic of manifolds which is an invariant of bordism classes. Such a homomorphism is called a (complex) R-genus.

Assume that the ring R does not have additive torsion. Then every R-genus φ is fully determined by the corresponding homomorphism $\Omega^U \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$, which we shall also denote by φ. A construction due to Hirzebruch describes homomorphisms $\varphi\colon \Omega^U \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$ by means of universal R-valued characteristic classes of special type.

Consider the evaluation homomorphism $e\colon \Omega^U \to H_*(BU)$ for tangential characteristic numbers. Then e is a monomorphism, and $e \otimes \mathbb{Q}\colon \Omega^U \otimes \mathbb{Q} \to H_*(BU;\mathbb{Q})$ is an isomorphism.

It follows that every homomorphism $\varphi\colon \Omega^U \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$ can be interpreted as an element of $\mathrm{Hom}_{\mathbb{Q}}(H_*(BU;\mathbb{Q}), R \otimes \mathbb{Q}) = H^*(BU;\mathbb{Q}) \otimes R$, or as a sequence of polynomials $\{K_i(c_1,\ldots,c_i),\ i \geqslant 0\}$, $\deg K_i = 2i$.

The fact that φ is a ring homomorphism imposes certain conditions on the sequence $\{K_i\}$. These conditions may be described as follows: an identity

$$1 + c_1 + c_2 + \cdots = (1 + c_1' + c_2' + \cdots) \cdot (1 + c_1'' + c_2'' + \cdots)$$

implies the identity

$$\sum_{n\geqslant 0} K_n(c_1,\ldots,c_n) = \sum_{i\geqslant 0} K_i(c_1',\ldots,c_i') \cdot \sum_{j\geqslant 0} K_j(c_1'',\ldots,c_j'').$$

A sequence $\mathcal{K} = \{K_i(c_1,\ldots,c_i), i \geqslant 0\}$ with $K_0 = 1$ satisfying the identities above is called a multiplicative Hirzebruch sequence.

Proposition

A multiplicative sequence $\mathcal{K}$ is completely determined by the series

$$Q(x) = 1 + q_1 x + q_2 x^2 + \cdots \in R \otimes \mathbb{Q}[[x]],$$

where $x = c_1$, and $q_i = K_i(1, 0, \ldots, 0)$; moreover, every series $Q(x)$ as above determines a multiplicative sequence.

Proof.

Indeed, by considering the identity

$$1 + c_1 + \cdots + c_n = (1 + x_1) \cdots (1 + x_n)$$

we obtain from the multiplicative property that

$$Q(x_1) \cdots Q(x_n) = 1 + K_1(c_1) + K_2(c_1, c_2) + \cdots \\ + K_n(c_1, \ldots, c_n) + K_{n+1}(c_1, \ldots, c_n, 0) + \cdots . \quad \square$$

Along with $Q(x)$ it is convenient to consider the series $f(x) \in R \otimes \mathbb{Q}[[x]] = x + \cdots$ given by the identity $Q(x) = \frac{x}{f(x)}$.

Given a genus $\varphi\colon \Omega^U \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$, the corresponding Hirzebruch sequence satisfies

$$K_n(c_1, \ldots, c_n) = \text{degree-}2n \text{ part of } \prod_{i=1}^{n} \frac{x_i}{f(x_i)} \in R \otimes \mathbb{Q}[[c_1, \ldots, c_n]].$$

We regard $\prod_{i=1}^{n} \frac{x_i}{f(x_i)}$ as a universal characteristic class of complex n-plane bundles. Then the value of φ on an $2n$-dimensional stably complex manifold M is given by

$$\varphi[M] = \Bigl(\prod_{i=1}^{n} \frac{x_i}{f(x_i)}(\mathcal{T}M)\Bigr)\langle M\rangle.$$

The Hirzebruch genus corresponding to a series $f(x) = x + \cdots \in R \otimes \mathbb{Q}[[x]]$ is the homomorphism $\varphi\colon \Omega^U \to R \otimes \mathbb{Q}$ given by the formula above.

Theorem

For every genus $\varphi\colon \Omega^U \to R$, the exponential of the formal group law $\varphi(F_U)$ is the series $f(x) \in R \otimes \mathbb{Q}[[x]]$ corresponding to φ.

This can be proved either directly, by appealing to the construction of geometric cobordisms, or indirectly, by calculating the values of φ on projective spaces and comparing to the formula for the logarithm of the formal group law.

Example

The universal genus maps a stably complex manifold M to its bordism class $[M] \in \Omega^U$ and therefore corresponds to the identity homomorphism $\varphi_U\colon \Omega^U \to \Omega^U$.

Its corresponding series $f_U(x)$ is the exponential of the universal formal group law of geometric cobordisms.

Example

We take $R = \mathbb{Z}$ in these examples.

1. The top Chern genus is given by $c[M] = c_n[M]$ for $[M] \in \Omega^U_{2n}$. We have $Q(x) = 1 + x$ and $f(x) = \frac{x}{1+x}$. Note that $c[M]$ is the Euler characteristic of M if $[M]$ is the cobordism class of an almost complex manifold M.

2. The L-genus $L[M]$ corresponds to the series $f(x) = \tanh(x)$. The L-genus coincides with the signature $\mathrm{sign}(M)$ of the manifold by the classical result of Hirzebruch. This can be seen by observing that $\mathrm{sign}(\mathbb{C}P^{2k}) = 1$ and $\mathrm{sign}(\mathbb{C}P^{2k+1}) = 0$ and calculating the functional inverse series $g(u)$ (the logarithm).

3. The Todd genus $\mathrm{td}[M]$ corresponds to the series $f(x) = 1 - e^{-x}$. The associated formal group law is given by $F(u, v) = u + v - uv$, so the Todd genus is integral on any complex bordism class.
The logarithm is given by $-\ln(1 - u) = \sum_{k \geqslant 1} \frac{u^k}{k}$, which implies $\mathrm{td}[\mathbb{C}P^k] = 1$ for any k. The Q-series is

$$Q(x) = \frac{x}{1-e^{-x}} = \sum_{k \geqslant 0} (-1)^k \frac{B_k}{k!} x^k.$$

Example

4. Another important example from the original work of Hirzebruch is given by the χ_y-genus. It corresponds to the series

$$f(x) = \frac{1 - e^{-x(1+y)}}{1 + ye^{-x(1+y)}},$$

where y is a parameter. Setting $y = -1$, $y = 0$ and $y = 1$ we get $c_n[M]$, the Todd genus $\mathrm{td}[M]$ and the L-genus $L[M] = \mathrm{sign}(M)$ respectively.

When working with graded rings, it is convenient to consider the 2-parameter homogeneous genus corresponding to

$$f(x) = \frac{e^{ax} - e^{bx}}{ae^{bx} - be^{ax}}, \quad \deg a = \deg b = -2.$$

It is called the $\chi_{a,b}$-genus.
One gets the original χ_y-genus by setting $a = y$, $b = -1$.

Complex oriented theories

A multiplicative generalised cohomology theory $X \mapsto h^*(X)$ is complex oriented if it has a choice of Euler class for every complex vector bundle. Such a choice is determined by a choice of an element $c_1^h \in \widetilde{h}^2(\mathbb{C}P^\infty)$ which restricts to 1 under the composite map

$$\widetilde{h}^2(\mathbb{C}P^\infty) \to \widetilde{h}^2(\mathbb{C}P^1) \cong h^0(pt).$$

c_1^h is called the universal first Chern class in the theory h^*.
For a complex line bundle ξ over X classified by a map $f\colon X \to BU(1)$, the first Chern class is defined by $c_1^h(\xi) = f^*(c_1^h) \in \widetilde{h}^2(X)$.

Examples of complex oriented theories include ordinary cohomology, complex K-theory and complex cobordism.

Given two complex line bundles ξ, η over X with $u = c_1^U(\xi)$ and $v = c_1^U(\eta)$,

$$F_h(u, v) = c_1^h(\xi \otimes \eta)$$

is a formal group law over $h^*(pt)$, as in the case of complex cobordism. The f.g.l. F_h is classified by a ring map $\Omega_U = U^*(pt) \to h^*(pt)$ (a genus), which extends to a transformation of cohomology theories $U^*(X) \to h^*(X)$.

Therefore, a complex oriented cohomology theory h^* defines a formal group law F_h and the corresponding genus $\Omega_U \to h^*(pt)$.

On the other hand, given a genus $\varphi\colon \Omega_U \to R$, one may try to define a cohomology theory by setting $h_\varphi^*(X) = U^*(X) \otimes_{\Omega_U} R$.

The functor $X \mapsto h_\varphi^*(X)$ is homotopy invariant and has the excision property. However, tensoring with R may fail to preserve exact sequences. A criterion for $h_\varphi^*(X)$ to be a cohomology theory is given next.

Define the *n-th power* in F_U as $[n](u) = F_U([n-1](u), u)$ and $[0](u) = 0$. For each prime p, write

$$[p](u) = pu + \cdots + t_1 u^p + \cdots + t_n u^{p^n} + \cdots ,$$

where $t_i \in \Omega_U^{-2(p^n-1)}$.

Theorem (Landweber Exact Functor Theorem)

In order that $U_*(X) \otimes_{\Omega_U} R$ be a homology theory, it suffices that for each prime p, the sequence $p, t_1, \ldots, t_n, \ldots$ of elements in Ω_U be R-regular. That is, it is required that the multiplication by p on R, and by t_n on $R/(pR + \cdots + t_{n-1}R)$ for $n \geqslant 1$, be injective.

If the condition above is satisfied for the homomorphism $\Omega^U \to h_*(pt)$ coming from a complex oriented homology theory h_*, the theory h_* is called *Landweber exact*. In this case, the canonical transformation

$$U_*(X) \otimes_{\Omega_U} h_*(pt) \longrightarrow h_*(X)$$

is an equivalence of homology theories.

Example

1. The Thom homomorphism $U^* \to H^*$ gives rise to the augmentation genus $\varepsilon\colon \Omega_U \to \mathbb{Z}$ sending each element of nonzero degree in Ω_U to zero. It corresponds to the series $f(x) = x$.
The ordinary cohomology theory H^* is not Landweber exact, because $\varepsilon(t_1) = 0$ and hence the multiplication by t_1 is zero on $\mathbb{Z}/p\mathbb{Z}$. Indeed, it is known that the identity $U_*(X) \otimes_{\Omega_U} \mathbb{Z} = H_*(X)$ does not hold in general.

On the other hand, the rational cohomology theory $H^*(X;\mathbb{Q})$ is Landweber exact; we have $U_*(X) \otimes_{\Omega_U} \mathbb{Q} = H_*(X;\mathbb{Q})$. The reason is that $\mathbb{Q}/p\mathbb{Q} = 0$.

2. The Todd genus $\mathrm{td}\colon \Omega_U \to \mathbb{Z}$ defines a Ω_U-module structure on $\mathbb{Z}$, which we denote by $\mathbb{Z}_{\mathrm{td}}$ for emphasis.
The p-th power in the corresponding formal group law is given by

$$[p]_{\mathrm{td}}(u) = 1 - (1-u)^p = pu + \cdots + u^p,$$

so t_1 acts identically on $\mathbb{Z}_{\mathrm{td}}/p\mathbb{Z}_{\mathrm{td}}$. Hence, Landweber's Theorem applies, and we get a cohomology theory $U^*(X) \otimes_{\Omega_U} \mathbb{Z}_{\mathrm{td}}$.

Example

On the other hand, there is a natural transformation

$$\mu_c \colon U^*(X) \to K^*(X)$$

from complex cobordism to complex K-theory (graded $\mod 2$), due to Conner and Floyd. Since $\mu_c \colon \Omega_U \to K^*(pt)$ is the same as the Todd genus, the above transformation factors through a transformation

$$\widetilde{\mu}_c \colon U^*(X) \otimes_{\Omega_U} \mathbb{Z}_{\mathrm{td}} \to K^*(X)$$

which is an equivalence by the uniqueness theorem for cohomology theories.

We therefore obtain the celebrated result of Conner and Floyd which states that complex cobordism determines complex K-theory.

Example

We can also obtain $\mathbb{Z}$-graded K-theory (which remembers the dimension of complex line bundles) by a similar procedure.

Then we have $K^*(pt) = \mathbb{Z}[\beta, \beta^{-1}]$ where $\beta = 1 - \bar{\eta}$ as the Bott element in $\widetilde{K}^0(\mathbb{C}P^1) = K^{-2}(pt)$, $\deg \beta = -2$.

We view $\mathbb{Z}[\beta, \beta^{-1}]$ as a graded Ω_U-module via the homomorphism $[M^{2n}] \mapsto \mathrm{td}[M^{2n}]\beta^n$. The corresponding formal group law has the p-th power is given by

$$[p]_\beta(u) = pu + \cdots + \beta^{p-1}u^p.$$

Landweber's Theorem applies because the multiplication by β^{p-1} is an isomorphism $\mathbb{Z}_p[\beta, \beta^{-1}] \to \mathbb{Z}_p[\beta, \beta^{-1}]$, and $\mathbb{Z}_p[\beta, \beta^{-1}]/(\beta^{p-1}) = 0$. We therefore obtain an equivalence of cohomology theories

$$U^*(X) \otimes_{\Omega_U} \mathbb{Z}[\beta, \beta^{-1}] \xrightarrow{\cong} K^*(X).$$

The conclusion is that both $\mathbb{Z}_2$- and $\mathbb{Z}$-graded versions of complex K-theory are Landweber exact.

Literature

[1] Victor Buchstaber and Taras Panov. *Toric Topology.* Mathematical Surveys and Monographs, vol. 204, Amer. Math. Soc., Providence, RI, 2015.