Lecture 1. Bordism and Cobordism: Basics

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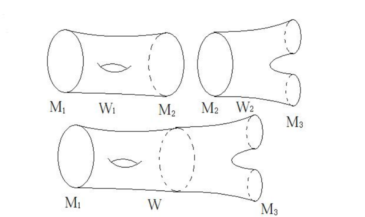
Bordism of manifolds

All manifolds M are smooth and closed, unless otherwise specified.

M_1 and M_2 are (co)bordant (notation: $M_1 \sim M_2$) if there exists a manifold W^{n+1} with boundary such that $\partial W^{n+1} = M_1 \sqcup M_2$.

Bordism $\sim$ is an equivalence relation.

Indeed, $M \sim M$ with $W = M \times [0,1]$;
$M_1 \sim M_2 \Rightarrow M_2 \sim M_1$ obvious;
$M_1 \sim M_2$ & $M_2 \sim M_3 \Longrightarrow M_1 \sim M_3$ as shown below.



Denote by $[M]$ the bordism equivalence class of M.
$\Omega^O_n = \{[M^n]\}$ the set of bordism classes of n-dimensional manifolds.

Ω^O_n is an abelian group: $[M_1] + [M_2] = [M_1 \sqcup M_2]$, $0 = [\varnothing]$.
We have $\partial(M \times I) = M \sqcup M$, so $2[M] = 0$ and Ω^O_n is a 2-torsion group.

Set $\Omega^O = \bigoplus_{n \geqslant 0} \Omega^O_n$. The product $[M_1] \times [M_2] = [M_1 \times M_2]$ makes Ω^O a graded commutative ring, the unoriented bordism ring.

For any space X the bordism relation can be extended to maps $M \to X$ of manifolds to X:
two maps $M_1 \to X$ and $M_2 \to X$ are bordant if there is a bordism W between M_1 and M_2 with a map $W \to X$ extending $M_1 \sqcup M_2 \to X$.

The set of bordism classes of maps $M \to X$ with $\dim M = n$ is the n-dimensional unoriented bordism group of X, denoted $O_n(X)$ (other notation: $\mathfrak{N}_n(X)$, $MO_n(X)$).

We have $O_n(pt) = \Omega_n^O$, where pt is a point.

There is a homomorphism $\Omega_m^O \times O_n(X) \to O_{m+n}(X)$ turning $O_*(X) = \bigoplus_{n\geqslant 0} O_n(X)$ into a graded Ω^O-module.

The assignment $X \mapsto O_*(X)$ defines a generalised homology theory, that is, it is functorial in X, homotopy invariant, has the excision property and exact sequences of pairs.

Pontryagin–Thom construction

$\xi\colon E \to X$ an n-dimensional real vector bundle over compact Hausdorff X.
$Th\,\xi = BE/SE$ the Thom space (a one-point compactification of E).

$\eta_k\colon\ EO(k) \to BO(k)$ be the universal vector k-plane bundle.
Following the original notation of Thom, we denote $MO(k) = Th\,\eta_k$.

$M^n \subset \mathbb{R}^{n+k}$ a submanifold with normal bundle ν.
The Pontryagin–Thom map $S^{n+k} \to Th\,\nu$ collapses the complement of a tubular neighbourhood of M^k in $\mathbb{R}^{n+k}$ to the basepoint of $Th\,\nu$.

Composing with the classifying map $\nu \to \eta_k$ obtain a map
$S^{n+k} \to Th\,\nu \to MO(k)$.

Conversely, given a map $f\colon S^{n+k} \to MO(k)$, we change it in its homotopy class so that it becomes transverse to the zero section $BO(k) \subset MO(k)$.
Then $M := f^{-1}(BO(k)) \subset S^{n+k}$ is an embedded n-manifold.

Changing $f\colon S^{n+k} \to MO(k)$ produces bordant manifolds M. The result is the Thom isomorphism

$$\Omega_n^O \cong \lim_{k\to\infty} \pi_{k+n}\bigl(MO(k)\bigr).$$

This generalises to bordism of maps $M \to X$ as

$$O_n(X) \cong \lim_{k\to\infty} \pi_{k+n}\bigl((X_+) \wedge MO(k)\bigr)$$

where $X_+ = X \sqcup pt$.

The cobordism groups of X are defined dually:

$$O^n(X) = \lim_{k\to\infty} \bigl[\Sigma^{k-n}(X_+), MO(k)\bigr].$$

It follows that $O^n(pt) = O_{-n}(pt)$. The graded ring Ω_O^* with $\Omega_O^{-n} := O^{-n}(pt) = \Omega_n^O$ is called the unoriented cobordism ring. It has nonzero elements only in nonpositively graded components.

Oriented bordism

The bordism relation may be extended to manifolds endowed with some additional structure, which leads to important bordism theories.

The simplest additional structure is an orientation. Two oriented n-dimensional manifolds M_1 and M_2 are oriented bordant if there is an oriented $(n+1)$-dimensional manifold W with boundary such that $\partial W = M_1 \sqcup \overline{M}_2$, where $\overline{M}_2$ denotes M_2 with the orientation reversed.

The oriented bordism groups Ω_n^{SO} and the oriented bordism ring $\Omega^{SO} = \bigoplus_{n \geqslant 0} \Omega_n^{SO}$ are defined accordingly.

Given an oriented manifold M, the manifold $M \times I$ has the canonical orientation such that $\partial(M \times I) = M \sqcup \overline{M}$. Hence, $-[M] = [\overline{M}]$ in Ω_n^{SO}. Unlike Ω_n^{O}, elements of Ω^{SO} generally do not have order 2.

Stably complex structures

A tangential stably complex structure on M is determined by a choice of an isomorphism

$$c_{\mathcal{T}}\colon \mathcal{T}M \oplus \underline{\mathbb{R}}^k \xrightarrow{\cong} \xi$$

between the stable tangent bundle and a complex vector bundle ξ over M.

Stably complex structures are equivalent if they differ by adding trivial complex summands and composing with isomorphisms of complex bundles. The equivalence class of $c_{\mathcal{T}}$ is the equivalence class of a lift

$$\begin{tikzcd}
 & BU(N) \arrow[d] \\
M \arrow[ur, dashed] \arrow[r] & BO(2N)
\end{tikzcd}$$

of the classifying map for $\mathcal{T}M$ to $BU(N)$ up to homotopy and stabilisation.

A tangential stably complex manifold is a pair $(M, c_{\mathcal{T}})$.

Example

The standard complex structure on $\mathbb{C}P^1$ is equivalent to the stably complex structure determined by the isomorphism

$$\mathcal{T}\mathbb{C}P^1 \oplus \underline{\mathbb{R}}^2 \xrightarrow{\cong} \bar{\eta} \oplus \bar{\eta}$$

where η is the tautological line bundle.

On the other hand, one can view $\mathbb{C}P^1$ as S^2 embedded into $\mathbb{R}^4 \cong \mathbb{C}^2$ with trivial normal bundle. We therefore have an isomorphism

$$\mathcal{T}\mathbb{C}P^1 \oplus \underline{\mathbb{R}}^2 \xrightarrow{\cong} \underline{\mathbb{C}}^2 \cong \bar{\eta} \oplus \eta$$

which determines a trivial stably complex structure on $\mathbb{C}P^1 \cong S^2$.

A normal stably complex structure on M is determined by a choice of a complex bundle structure on the normal bundle $\nu(M)$ of an embedding $M \hookrightarrow \mathbb{R}^N$. Tangential and normal stably complex structures on M determine each other by means of the isomorphism $\mathcal{T}M \oplus \nu(M) \cong \underline{\mathbb{R}}^N$.

Complex bordism

The bordism relation can be defined between stably complex manifolds. The n-dimensional complex bordism group Ω_n^U consists of bordism classes of $[M^n, c_{\mathcal{T}}]$ of stably complex n-manifolds.

The opposite element to $[M, c_{\mathcal{T}}] \in \Omega_n^U$ may be represented by the same M with the stably complex structure determined by the isomorphism

$$\mathcal{T}M \oplus \underline{\mathbb{R}}^k \oplus \underline{\mathbb{C}} \xrightarrow{c_{\mathcal{T}} \oplus \tau} \xi \oplus \underline{\mathbb{C}}$$

where $\tau \colon \mathbb{C} \to \mathbb{C}$ is the complex conjugation.

The direct product of stably complex manifolds turns $\Omega^U = \bigoplus_{n \geqslant 0} \Omega_n^U$ into a graded ring, the complex bordism ring.

The complex bordism groups $U_n(X)$ and cobordism groups $U^n(X)$ may be defined homotopically as

$$U_n(X) = \lim_{k\to\infty} \pi_{2k+n}\big((X_+) \wedge MU(k)\big),$$
$$U^n(X) = \lim_{k\to\infty} \big[\Sigma^{2k-n}(X_+), MU(k)\big],$$

where $MU(k)$ is the Thom space of the universal complex k-plane bundle over $BU(k)$. Here the direct limit uses the maps $\Sigma^2 MU(k) \to MU(k+1)$.

These groups are Ω_*^U-modules and give rise to a multiplicative (co)homology theory. In particular, $U^*(X) = \prod_n U^n(X)$ is a graded ring.

Ω_U^* with $\Omega_U^n = \Omega_{-n}^U = U^n(pt)$ is the complex cobordism ring; it has nontrivial elements only in nonpositively graded components.

Characteristic classes and numbers

The classifying space $BU(m)$ of complex m-plane has

$$H^*(BU(m)) = \mathbb{Z}[c_1, \ldots, c_m], \quad \deg c_i = 2i.$$

$c_i \in H^{2i}(BU(m))$ the universal Chern characteristic classes.

ξ a complex m-plane bundle ξ over X classified by a map $f\colon X \to BU(m)$.
The ith Chern characteristic class of ξ is $c_i(\xi) = f^*(c_i) \in H^{2i}(X)$.
The total Chern class of ξ is

$$c(\xi) = 1 + c_1(\xi) + \cdots + c_n(\xi).$$

Chern classes are determined uniquely by their functoriality,
Whitney sum formula $c(\xi \oplus \eta) = c(\xi)c(\eta)$
and normalisation: $c_1(\bar{\eta}) = u \in H^2(\mathbb{C}P^n)$ dual to $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$.

$(M, c_{\mathcal{T}})$ a $2n$-dimensional stably complex manifold.

Given a polynomial in Chern classes $c \in \mathbb{Z}[c_1, \ldots, c_n]$ of degree $2n$, the corresponding tangential Chern number $c[M] \in \mathbb{Z}$ is the result of pairing of $c(\mathcal{T}M) \in H^{2n}(M)$ with the fundamental class $\langle M \rangle \in H_{2n}(M)$. The number $c[M]$ depends only on the complex bordism class of M.

Tangential Stiefel–Whitney numbers $w[M] \in \mathbb{Z}_2$ of n-manifolds and Pontryagin numbers $p[M] \in \mathbb{Z}$ of oriented $4n$-manifolds are defined similarly; they are unoriented and oriented bordism invariants, respectively.

$M \hookrightarrow \mathbb{R}^N$ an embedding with a fixed complex structure in the normal bundle ν, classified by a map $g \colon M \to BU(k)$.
The normal Chern number $\bar{c}[M]$ corresponding to
$c \in H^*(BU(k)) = \mathbb{Z}[c_1, \ldots, c_k]$ is defined as $\bar{c}[M] := (g^*c)\langle M \rangle$.
Normal Stiefel–Whitney and Pontryagin numbers are defined similarly.
As we have $\mathcal{T}M \oplus \nu = \underline{\mathbb{R}}^N$, the tangential and normal characteristic numbers determine each other.

We proceed to describe two different ways of encoding characteristic classes and numbers by integer vectors.

We consider nonnegative integer vectors $\omega = (i_1, \ldots, i_n)$ and denote

$$\|\omega\| = \sum_{k=1}^{n} k\, i_k.$$

Vectors $\omega = (i_1, \ldots, i_n)$ encode partitions of $\|\omega\|$ into a sum of positive integers; the kth component i_k of ω is the number of summands k.

To each $\omega = (i_1, \ldots, i_n)$ one assigns the universal characteristic class

$$c_\omega = c_1^{i_1} c_2^{i_2} \cdots c_n^{i_n} \in H^{2\|\omega\|}(BU(n)).$$

Another way of assigning a characteristic class to an integer vector is described next.

ξ be a complex n-plane bundle over a manifold M. Write formally

$$c(\xi) = 1 + c_1(\xi) + \cdots + c_n(\xi) = (1 + t_1) \cdots (1 + t_n),$$

so that $c_i(\xi) = \sigma_i(t_1, \ldots, t_n)$ is the ith elementary symmetric function.

These indeterminates $t_1, \ldots, t_n$ acquire a geometric meaning if ξ is a sum $\xi_1 \oplus \cdots \oplus \xi_n$ of line bundles; then $t_j = c_1(\xi_j)$, $1 \leqslant j \leqslant n$.
The canonical fibre bundle $BT^n \to BU(n)$ induces an embedding of $H^*(BU(n)) = \mathbb{Z}[c_1, \ldots, c_n]$ in $H^*(BT^n) = \mathbb{Z}[t_1, \ldots, t_n]$.

Given $\omega = (i_1, \ldots, i_n)$, define the universal symmetric polynomial

$$P_\omega(t_1, \ldots, t_n) = t_1 \cdots t_{i_1} t_{i_1+1}^2 \cdots t_{i_1+i_2}^2 \cdots t_{i_1+i_2+\cdots+i_n}^n + \cdots$$

as the smallest symmetric polynomial containing the monomial above.
Express P_ω via the elementary symmetric functions and substitute the Chern classes. We obtain a universal characteristic class

$$s_\omega = s_\omega(c_1, \ldots, c_n) = P_\omega(t_1, \ldots, t_n) = \in H^{2\|\omega\|}(BU(n)).$$

Example

Let $\omega = (k, 0, \dots, 0)$ be the partition into a sum of k units. Then

$$c_{(k,0,\dots,0)} = c_1^k, \quad P_{(k,0,\dots,0)} = \sigma_k, \quad s_{(k,0,\dots,0)} = c_k.$$

Now let $\omega = (0, \dots, 0, 1, 0, \dots, 0)$ with unit at the kth position. Then

$$c_{(0,\dots,0,1,0,\dots,0)} = c_k, \quad P_{(0,\dots,0,1,0,\dots,0)} = t_1^k + \dots + t_n^k.$$

Traditionally, the characteristic class $s_{(0,\dots,0,1,0,\dots,0)} \in H^{2k}(BU(n))$ is denoted simply by s_k. For example,

$$s_1 = c_1, \quad s_2 = c_1^2 - 2c_2, \quad s_3 = c_1^3 - 3c_1c_2 + 3c_3.$$

Theorem

$$s_\omega(\xi \oplus \eta) = \sum_{\omega=\omega'+\omega''} s_{\omega'}(\xi)\, s_{\omega''}(\eta)$$

Proposition

(a) $s_k(\xi) = 0$ *if* ξ *is a bundle over* X *and* $\dim X < 2k$*;*

(b) $s_k(\xi \oplus \eta) = s_k(\xi) + s_k(\eta)$*;*

(c) $s_k(\xi) = c_1(\xi)^k$ *if* ξ *is a line bundle.*

For a stably complex $2n$-dimensional manifold $(M, c_{\mathcal{T}})$, the characteristic numbers are defined by evaluation:

$$c_\omega[M] = c_\omega(\mathcal{T}M)\langle M\rangle, \quad s_\omega[M] = s_\omega(\mathcal{T}M)\langle M\rangle, \quad s_n[M] = s_n(\mathcal{T}M)\langle M\rangle.$$

Corollary

If a bordism class $[M] \in \Omega^U_{2n}$ *decomposes as* $[M_1] \times [M_2]$ *where* $\dim M_1 > 0$ *and* $\dim M_2 > 0$*, then* $s_n[M] = 0$*.*

$BU = \lim_{n\to\infty} BU(n)$ has $H^*(BU) \cong \mathbb{Z}[[c_1, c_2, \ldots]]$, $\deg c_k = 2k$.

The evaluation homomorphism

$$e\colon \Omega^U \to H_*(BU) \subset \mathrm{Hom}(H^*(BU), \mathbb{Z}), \qquad ([M], c) \mapsto c[M].$$

Proposition

The evaluation homomorphism $\bar{e}$ for normal Chern numbers coincides with the Hurewicz homomorphism in complex cobordism:

$$\Omega^U_{2n} = \lim_{k\to\infty} \pi_{2k+2n}(MU(k)) \xrightarrow{\text{Hurewicz}} \lim_{k\to\infty} H_{2k+2n}(MU(k))$$
$$\xrightarrow{\text{Thom}} \lim_{k\to\infty} H_{2n}(BU(k)) = H_{2n}(BU),$$

Theorem (Thom)

$$e \otimes \mathbb{Q}\colon \Omega^U \otimes \mathbb{Q} \xrightarrow{\cong} H_*(BU; \mathbb{Q}).$$

Therefore, for any set of integers $\{k_\omega\}$, there exists a stably complex manifold M such that $s_\omega[M] = Nk_\omega$ for some fixed $N \in \mathbb{Z}$ and any ω.

Structure results

The theory of unoriented (co)bordism was the first to be completed: the coefficient ring Ω^O was calculated by Thom, and the bordism groups $O_*(X)$ of cell complexes X were reduced to homology groups of X with coefficients in Ω^O.

Theorem (Thom, Conner–Floyd)

(a) *Two manifolds are unorientedly bordant if and only if they have identical sets of Stiefel–Whitney characteristic numbers.*

(b) *Ω^O is a polynomial ring over $\mathbb{Z}_2$ with one generator a_i in every positive dimension $i \neq 2^k - 1$.*

(c) *For every cell complex X the module $O_*(X)$ is a free graded Ω^O-module isomorphic to $H_*(X;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Omega^O$.*

Parts (a) and (b) were done by Thom in 1954. Part (c) was first formulated by Conner and Floyd in 1964; it also follows from the results of Thom.

Theorem (Milnor, Novikov)

(a) *$\Omega^U \otimes \mathbb{Q}$ is a polynomial ring over $\mathbb{Q}$ generated by the bordism classes of complex projective spaces $\mathbb{C}P^i$, $i \geqslant 1$.*

(b) *Two stably complex manifolds are bordant if and only if they have identical sets of Chern characteristic numbers.*

(c) *$\Omega^U \cong \mathbb{Z}[a_1, a_2, \ldots]$, a polynomial ring over $\mathbb{Z}$ with one generator a_i in every even dimension $2i$, where $i \geqslant 1$.*

Part (a) follows from the results of Thom. Part (c) is the most difficult one; it was done by Novikov (1960, 1962) using the Adams spectral sequence and structure theory of Hopf algebras, and by Milnor (1960, unpublished). Another more geometric proof was given by Stong in 1965.

Note that $U_*(X)$ is not a free Ω^U-module in general, unlike the unoriented bordism case. The theory of complex (co)bordism is much richer than its unoriented analogue, and at the same time is not as complicated as oriented bordism or other bordism theories with additional structure.

The calculation of the oriented bordism ring was completed in 1960 by Novikov (ring structure modulo torsion and odd torsion) and Wall (even torsion), with important earlier contributions made by Rokhlin, Averbuch, and Milnor. Unlike complex bordism, the ring Ω^{SO} has additive torsion.

Theorem (Rokhlin, Averbuch, Milnor, Novikov, Wall)

(a) *$\Omega^{SO} \otimes \mathbb{Q}$ is a polynomial ring over $\mathbb{Q}$ generated by the bordism classes of complex projective spaces $\mathbb{C}P^{2i}$, $i \geqslant 1$.*

(b) *The subring $\mathrm{Tors} \subset \Omega^{SO}$ of torsion elements contains only elements of order 2. The quotient $\Omega^{SO}/\mathrm{Tors}$ is a polynomial ring over $\mathbb{Z}$ with one generator a_i in every dimension $4i$, where $i \geqslant 1$.*

(c) *Two oriented manifolds are bordant if and only if they have identical sets of Pontryagin and Stiefel–Whitney characteristic numbers.*

Construction (geometric cobordisms)

For any cell complex X, have $H^2(X) = [X, \mathbb{C}P^\infty]$. Since $\mathbb{C}P^\infty = MU(1)$, every element $x \in H^2(X)$ determines a cobordism class $u_x \in U^2(X)$, a geometric cobordism. Hence, $H^2(X) \subset U^2(X)$ (a subset, not a subgroup!)

When X is a manifold, each $u_x \in U^2(X)$ corresponds to a submanifold $M \subset X$ of codimension 2 with a complex structure on the normal bundle.

Indeed, $x \in H^2(X)$ corresponds to a homotopy class of $f_x \colon X \to \mathbb{C}P^\infty$. May assume $f_x(X)$ is transverse to a hyperplane $H \subset \mathbb{C}P^N \subset \mathbb{C}P^\infty$. Then $M_x = f_x^{-1}(H)$ is a codimension-2 submanifold in X.
A homotopy of f_x gives a bordism of $M_x \to X$.

Conversely, given an embedding $i \colon M \subset X$ as above, the composite $X \to Th(\nu) \to MU(1) = \mathbb{C}P^\infty$ of the Pontryagin–Thom collapse map and the classifying map for ν defines an element $x_M \in H^2(X)$, and therefore a geometric cobordism.

If X is oriented, then $i_*\langle M \rangle \in H_*(X)$ is Poincaré dual to $x_M \in H^2(X)$.

Ring generators for Ω^U

As we have seen, the characteristic number s_n vanishes on decomposable elements of Ω^U. Furthermore, this characteristic number detects indecomposables that may be chosen as polynomial generators:

Theorem

A bordism class $[M] \in \Omega^U_{2n}$ may be chosen as a polynomial generator a_n of the ring Ω^U if and only if

$$s_n[M] = \begin{cases} \pm 1 & \text{if } n \neq p^k - 1 \text{ for any prime } p; \\ \pm p & \text{if } n = p^k - 1 \text{ for some prime } p. \end{cases}$$

There is no universal description of manifolds representing the polynomial generators $a_n \in \Omega^U$. On the other hand, there is a particularly nice family of manifolds whose bordism classes generate the whole ring Ω^U. This family is redundant though, so there are algebraic relations between their bordism classes.

Construction (Milnor hypersurfaces)

The Milnor hypersurface in $\mathbb{C}P^i \times \mathbb{C}P^j$ $(0 \leqslant i \leqslant j)$ is

$$H_{ij} = \{(z_0 : \dots : z_i) \times (w_0 : \dots : w_j) \in \mathbb{C}P^i \times \mathbb{C}P^j : z_0 w_0 + \dots + z_i w_i = 0\}$$

Note that $H_{0j} \cong \mathbb{C}P^{j-1}$.

More intrinsically, H_{ij} is a hyperplane section of the Segre embedding

$$\sigma \colon \mathbb{C}P^i \times \mathbb{C}P^j \to \mathbb{C}P^{(i+1)(j+1)-1},$$
$$(z_0 : \dots : z_i) \times (w_0 : \dots : w_j) \mapsto (z_0 w_0 : z_0 w_1 : \dots : z_k w_l : \dots : z_i w_j),$$

Also, H_{ij} may be identified with the set of pairs (ℓ, α), where ℓ is a line in $\mathbb{C}^{i+1}$ and α is a hyperplane in $\mathbb{C}^{j+1}$ containing ℓ.
In particular, $H_{22} = Fl(\mathbb{C}^3)$, the flag manifold.
The projection $H_{ij} \to \mathbb{C}P^i$, $(\ell, \alpha) \mapsto \ell$, is a fibre bundle with fibre $\mathbb{C}P^{j-1}$.

Denote by p_1 and p_2 the projections of $\mathbb{C}P^i \times \mathbb{C}P^j$ onto its factors. Then

$$H^*(\mathbb{C}P^i \times \mathbb{C}P^j) = \mathbb{Z}[x, y]/(x^{i+1} = 0,\ y^{j+1} = 0)$$

where $x = p_1^* c_1(\bar\eta)$, $y = p_2^* c_1(\bar\eta)$, and η the tautological bundle.

Proposition

H_{ij} represents the geometric cobordism in $\mathbb{C}P^i \times \mathbb{C}P^j$ corresponding $x + y \in H^2(\mathbb{C}P^i \times \mathbb{C}P^j)$. In particular, the image of the fundamental class $\langle H_{ij}\rangle$ in $H_{2(i+j-1)}(\mathbb{C}P^i \times \mathbb{C}P^j)$ is Poincaré dual to $x + y$.

Proof.

We have $x + y = c_1(p_1^*(\bar\eta) \otimes p_2^*(\bar\eta))$. The classifying map for $p_1^*(\bar\eta) \otimes p_2^*(\bar\eta)$ is the Segre embedding $\sigma\colon \mathbb{C}P^i \times \mathbb{C}P^j \to \mathbb{C}P^{(i+1)(j+1)-1} \to \mathbb{C}P^\infty$.
The codimension-2 submanifold in $\mathbb{C}P^i \times \mathbb{C}P^j$ corresponding to $x + y$ is the preimage $\sigma^{-1}(H)$ of a generally positioned hyperplane in $\mathbb{C}P^{(i+1)(j+1)-1}$. The Milnor hypersurface H_{ij} is exactly $\sigma^{-1}(H)$ for one such hyperplane H. □

Lemma

$$s_{i+j-1}[H_{ij}] = \begin{cases} j & \textit{if } \ i = 0; \\ -\binom{i+j}{i} & \textit{if } \ i > 1. \end{cases}$$

Proof.

The stably complex structure on $H_{0j} = \mathbb{C}P^{j-1}$ comes from the isomorphism $\mathcal{T}(\mathbb{C}P^{j-1}) \oplus \mathbb{C} \cong \bar\eta \oplus \cdots \oplus \bar\eta$ (j summands) and $x = c_1(\bar\eta)$, so have

$$s_{j-1}[\mathbb{C}P^{j-1}] = jx^{j-1}\langle \mathbb{C}P^{j-1}\rangle = j.$$

Denote by ν the normal bundle of $\iota\colon H_{ij} \hookrightarrow \mathbb{C}P^i \times \mathbb{C}P^j$. Then

$$\mathcal{T}(H_{ij}) \oplus \nu = \iota^*(\mathcal{T}(\mathbb{C}P^i \times \mathbb{C}P^j)).$$

We have $s_{i+j-1}(\nu) = \iota^*(x+y)^{i+j-1}$ and
$s_{i+j-1}(\mathcal{T}(\mathbb{C}P^i \times \mathbb{C}P^j)) = (i+1)x^{i+j-1} + (j+1)y^{i+j-1} = 0$ for $i > 1$, so

$$\begin{aligned} s_{i+j-1}[H_{ij}] &= -s_{i+j-1}(\nu)\langle H_{ij}\rangle = -\iota^*(x+y)^{i+j-1}\langle H_{ij}\rangle \\ &= -(x+y)^{i+j}\langle \mathbb{C}P^i \times \mathbb{C}P^j\rangle = -\binom{i+j}{i}. \qquad \square \end{aligned}$$

Theorem

The bordism classes $\{[H_{ij}], 0 \leqslant i \leqslant j\}$ *generate the ring* Ω^U.

Proof.

$$\text{g.c.d.}\Big(\binom{n+1}{i},\ 1 \leqslant i \leqslant n\Big) = \begin{cases} p & \text{if } n = p^k - 1, \\ 1 & \text{otherwise.} \end{cases}$$

Now the previous calculation of $s_{i+j-1}[H_{ij}]$ implies that a certain integer linear combination of bordism classes $[H_{ij}]$ with $i + j = n + 1$ has s_{i+j-1} equal p or 1, as needed for the polynomial generator a_n. □

Example

- $\Omega^U_2 = \mathbb{Z}$, generated by $[\mathbb{C}P^1]$, as $1 = 2^1 - 1$ and $s_1[\mathbb{C}P^1] = 2$;
- $\Omega^U_4 = \mathbb{Z} \oplus \mathbb{Z}$, generated by $[\mathbb{C}P^1 \times \mathbb{C}P^1]$ and $[\mathbb{C}P^2]$, as $2 = 3^1 - 1$ and $s_2[\mathbb{C}P^2] = 3$;
- $[\mathbb{C}P^3]$ cannot be taken as the polynomial generator $a_3 \in \Omega^U_6$, since $s_3[\mathbb{C}P^3] = 4$, while $s_3(a_3) = \pm 2$. We have $a_3 = [H_{22}] + [\mathbb{C}P^3]$.

Theorem (Milnor)

Every bordism class $x \in \Omega_n^U$ with $n > 0$ contains a nonsingular algebraic variety (not necessarily connected).

The proof of this fact uses a construction of a (possibly disconnected) algebraic variety representing the class $-[M]$ for any bordism class $[M] \in \Omega_n^U$ of $2n$-dimensional manifold.

Problem (Hirzebruch)

Describe the set of bordism classes in Ω^U containing connected nonsingular algebraic varieties.

Example

Every class $k[\mathbb{C}P^1] \in \Omega_2^U$ contains a nonsingular algebraic variety, namely, a disjoint union of k copies of $\mathbb{C}P^1$ for $k > 0$ and a Riemann surface of genus $(1-k)$ for $k \leqslant 0$.
Connected algebraic varieties are contained only in $k[\mathbb{C}P^1]$ with $k \leqslant 1$.

Literature

[1] Robert E. Stong. *Notes on Cobordism Theory*. Math. Notes, 7. Princeton Univ. Press, Princeton, NJ, 1968.

[2] Victor Buchstaber and Taras Panov. *Toric Topology.* Mathematical Surveys and Monographs, vol. 204, Amer. Math. Soc., Providence, RI, 2015.