Projective toric generators in the unitary cobordism ring

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Glances@Manifolds II

Kraków

August 10, 2016

Introduction. Complex cobordisms

Definition (Stably complex structure)

(M, ξ, i) with

- M — n-dimensional compact manifold
- $\xi \to M$ — complex vector bundle of rank m
- $i\colon TM \oplus \mathbb{R}^{2m-n} \simeq \xi$

Definition (Complex bordism relation)

(M_1^n, ξ_1, i_1) is bordant to (M_2^n, ξ_2, i_2) if there exists (W^{n+1}, ξ, i) s.t. $\partial W = M_1 \cup M_2$ and ξ induces ξ_1 and $-\xi_2$ on M_1 and M_2.

Definition (Complex cobordism ring)

$\Omega_U^* := \{(M, \xi, i)\}/\sim$ with disjoint union $\sqcup$ as addition and cross product $\times$ as multiplication.

Introduction. Complex cobordisms

For a stably complex manifold let $c(M)$ be the total Chern class of M:

$$c(M) := 1 + c_1(\xi) + \cdots + c_n(\xi) \in H^*(M, Z)$$

For $I = \{i_1, \ldots, i_t\}$ — a partition of n, consider cohomology class $c_{i_1}(M) \ldots c_{i_t}(M) \in H^{2n}(M)$ and define characteristic number

$$c_I(M) := \langle c_{i_1}(M) \ldots c_{i_t}(M), [M] \rangle \in \mathbb{Z}$$

Theorem (Milnor, Novikov, 1960s)

- $(M_1, \xi_1, i_1) \sim (M_2, \xi_2, i_2)$ *iff* $c_I(M_1) = c_I(M_2)$ *for all partitions* I.
- $\Omega_U^* \simeq \mathbb{Z}[a_1, a_2, \ldots]$ *with* $\deg a_i = 2i$.
- (M^{2n}, ξ, i) *can be taken as* a_n *iff*

$$s_n(M^{2n}) = \begin{cases} \pm 1, & n \neq p^k - 1 \quad \text{for any prime number } p; \\ \pm p, & n = p^k - 1 \quad \text{for some prime number } p, \end{cases}$$

where $s_n = p_n(c_1, \ldots, c_n)$ *— Newton polynomial.*

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Example

For a projective space $\mathbb{C}P^n$ we have $s_n(\mathbb{C}P^n) = n + 1$, so it can be taken as a generator iff $n + 1$ is a prime number.

Introduction. Generators and representatives in Ω_U^*

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4. [Johnston, 2004] connected algebraic generators
5. [Wilfong, 2014] connected toric generators in some dimensions (n — odd or $n = p^k - 1$)

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6 [U., Solomadin, 2016] connected toric generators in all dimensions

Introduction. Toric varieties

Definition (Toric varieties)

Smooth complete projective variety X over $\mathbb{C}$ is toric if $(\mathbb{C}^*)^{\dim_{\mathbb{C}} X}$ acts on X with dense open orbit.

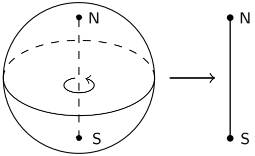
Example

Projective spaces $\mathbb{C}P^n$, products, Hirzebruch surfaces etc.

Construction

Every n-dimensional Delzant polytope P defines an n-dimensional smooth complete projective variety X_P such that $X_P/(S^1)^n \simeq P$.

$X_P = \mathbb{C}P^1$



$P = [a, b] \subset \mathbb{R}$

Introduction. Blow-ups

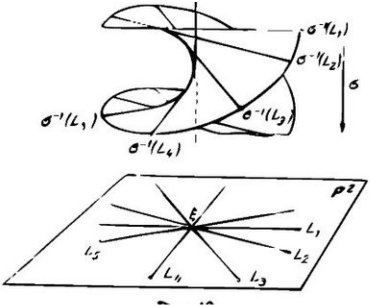
Definition (Blow up of a point in a disk)

For a unit disk $D \subset \mathbb{C}^n$ blow up is

$$Bl_0 D = \{(z, L) \in D \times \mathbb{C}P^n | z \in L\}.$$

Blow-up of a subdisk D^k in D^n:

$$Bl_{D^k} D^n = D^k \times Bl_0 D^{n-k}$$



Similarly to the above **local picture** for a submanfiold $Z \subset X$ we define blow-up $Bl_Z X \to X$

Proof. Overview

Theorem

There exist smooth projective toric varieties $\{X^n\}_{n=1}^{\infty}$, $\dim_{\mathbb{C}} X_n = n$ *such that*

$$\Omega_*^U = \mathbb{Z}[[X_1], [X_2], \ldots].$$

- algebraic topology
- combinatorics
- elementary number theory
- tedious computations

Let n be an even number such that $n+1$ is not a power of a prime (these are dimensions not covered by Wilfong result)

Goal: find toric X_P such that $s_n(X_P) = \pm 1$

Idea: (1) start with some toric manifold X_0, (2) apply a (wisely-chosen) sequence of equivariant blowups $X_n = Bl_{Z_{n-1}} X_{n-1}$ and (3) keep track of s_n.

Proof. Step 1. Blow-ups in Ω_U^*

Change of s_n under blow-up $Bl_Z X \to X$ depends only on the normal bundle $\nu(Z \subset X)$:

Theorem (Hitchin 1974)

For a complex manifold X and submanifold $Z \subset X$

$$[Bl_Z X] - [X] = -[\mathbb{P}(\nu(Z \subset X) \oplus \overline{\mathbb{C}})] \in \Omega_U^*,$$

where $\mathbb{P}(\nu(Z \subset X) \oplus \overline{\mathbb{C}})$ is endowed with a non-standard stably compex structure.*

* the standard complex structure on $\mathbb{C}P^n$ is given by:

$$T\mathbb{C}P^n \oplus \mathbb{R}^2 \simeq \underbrace{\mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1)}_{n+1}$$

in the non standard case

$$T\mathbb{C}P^n \oplus \mathbb{R}^2 \simeq \underbrace{\mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1)}_{n} \oplus \mathcal{O}(-1)$$

Proof. Step 2. Equivariant modifications

Definition (Operation B_k)

Fix X, $\dim_{\mathbb{C}} X = n$. For $k = 0, \dots, n-2$ define a tower of blow-ups:

$$B_k(X) := Bl_{Z^k} Bl_x X \to Bl_x X \to X,$$

where $x \in X$, $Z_k \simeq \mathbb{C}P^k$ is a projective subspace of the exceptional divisor $E \simeq \mathbb{C}P^{n-1}$, $E = \pi^{-1}x$ of $\pi \colon Bl_x X \to X$.

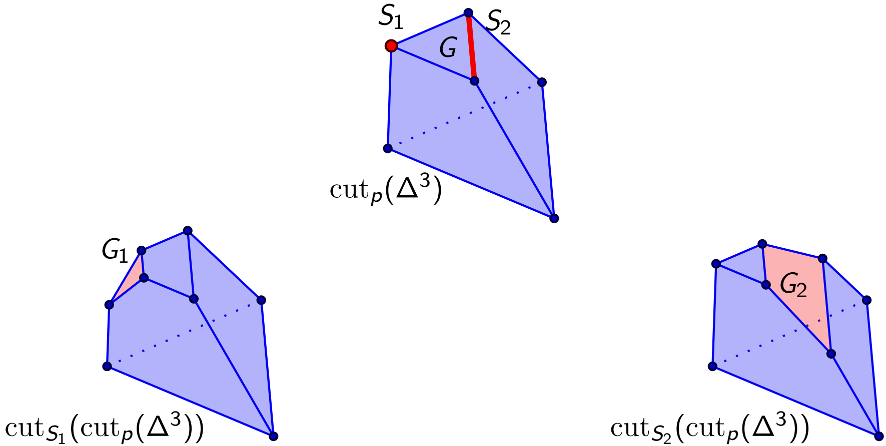
If $x \in X$ is a fixed point of a toric manifold, then $B_k(X)$ is also a toric manifold (with appropriate choice of Z^k).

Definition

$$s_{n,k} := s_n(B_k(X)) - s_n(X).$$

Proof. Step 2. Equivariant modifications

If X_P corresponds to a polytope P, then $B_k(X_P)$ corresponds to a polytope, obtained from P by two face cuts, e.g. case of $\mathbb{C}P^3$:



Proof. Step 3. Number theoretical lemma

Lemma

For even n such that $n+1$ is not a power of a prime $gcd\{s_{n,k}\}_{k=0}^{n-2} = 1$

Two ingredients:

- long and boring computations of $s_{n,k}$ (Leray-Hirsch theorem)
- elementary number theory (Lucas theorem)

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$$s_{n,k} = s_n(B_k(X)) - s_n(X) =$$

$$= -\left((n-k-1)(2^{k+1}-1) + \sum_{i=0}^{k}(-1)^i \left(2^i + (-1)^n 2^{k-i}\right)\binom{n-1}{i} + n + (-1)^n\right)$$

Proof. Step 4. Frobenius problem

Lemma (Frobenius problem)

There exists $N > 0$ such that for any $N' \geq N$ there exist non-negative $a_0, \ldots, a_{n-2} \in \mathbb{Z}$ s.t.: $\sum_{i=0}^{n-2} a_i s_{n,i} = -N'$.

Lemma

For any N there exists a toric variety X with $s_n(X) > N + 1$.

Proof.

Take the projectivization of the split bundle over $\mathbb{C}P^1 \times \mathbb{C}P^1$:

$$\xi = \pi_1^* \mathcal{O}(1) \oplus \pi_2^* \mathcal{O}(a) \oplus \mathbb{C}^{n-3}.$$

Then $s_n(\mathbb{P}(\xi)) = a(n+1)$. □

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Lemma

For any N there exists toric variety with $s_n(X) > N + 1$.

Proof of the main result.

1. Find N for the Frobenius problem.
2. Take X_P from the second lemma.
3. Find $a_0, \ldots, a_{n-2}$ from the Frobenius problem for $N' = s_n(X) - 1$.
4. Apply to X_P equivariant modifications: a_0 times B_0, a_1 times B_1,...,a_{n-2} times B_{n-2}.
5. Resulting X_Q has $s_n(X_Q) = s_n(X_Q) + \sum_i a_i s_{n,i} = 1$

□

Application

Definition (Hirzebruch genera)

Let R be a ring without torsion. R-genus is a ring homomorphism $\phi\colon \Omega_U^* \to R$.

Example

Euler characteristic χ, signature L, Todd genus T are $\mathbb{Z}$-genera
2-parametric Todd genus $\chi_{a,b}$ — $\mathbb{Z}[a,b]$-genus, $\hat{A}$-genus — $\mathbb{Q}$-genus.

Definition

R-genus ϕ is said to be combinatorially rigid if for any 2 toric varieties X_P and X_Q with $P \simeq_{comb} Q$

$$\phi(X_P) = \phi(X_Q).$$

Application

Proposition

2-parametric Todd genus $\chi_{a,b}$ is combinatorially rigid.

Proof.

For a complex manifold M the value $\chi_{a,b}(M)$ is determined by $h^{p,q}(M) = \dim_{\mathbb{C}} H^q(M, \Omega^p)$.
For a toric variety X_P by the computations due to Danilov the Hodge numbers $h^{p,q}$ depend only on combinatorics of P. □

From now on we consider $\chi_{a,b}$ as a homomorphism onto $Sym(a, b)$.

Application

Theorem (Characterization of combinatorially rigid genera)

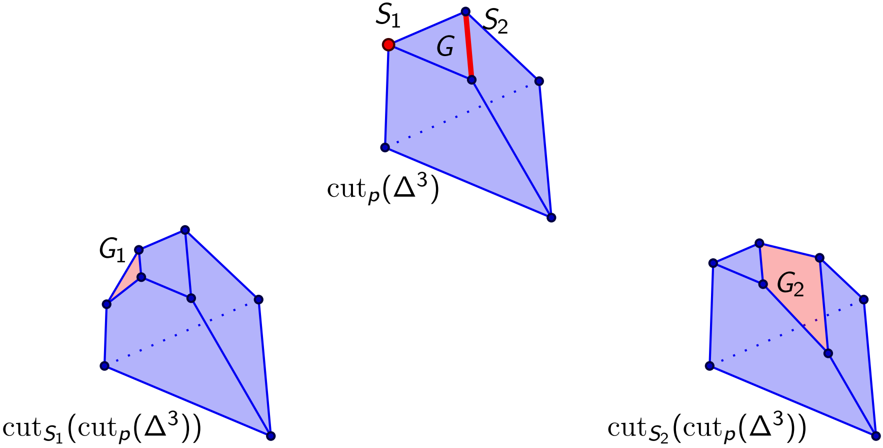
For any combinatorially rigid $\phi\colon \Omega^*_U \to R$ *there exists a transformation* $f\colon Sym(a,b) \to R$ *such that:*

$$\phi = f \circ \chi_{a,b}.$$

Proof.

For the proof it suffices to construct pairs X^n_P and X^n_Q with combinatorially equivalent P, Q but different $s_n(X_P), s_n(X_Q)$ in every dimension $n \geq 3$. $B_0(\mathbb{C}P^n)$ and $B_{n-2}(\mathbb{C}P^n)$ turn out to work![a] □

[a] In fact, underlying polytopes of $B_k(X^n_P)$ and $B_{n-k-2}(X^n_P)$ are always combinatorially equivalent.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Conclusion

We presented constructuve method for producing toric generators of Ω_U^*. However it is highly inefficient![1]

Problem. Find projective toric polynomial generators of the ring Ω_*^U with moment polytopes having:

a) the least number of vertices;

b) the least number of facets;

c) the least number of faces of all dimensions.

Problem. Find “toric” (combinatorial) characterization of other important Hirzebruch genera.

[1] Naive straightforward implementation for $n = 14$ requires $> 33K$ of modifications, each increasing the number of facets and vertices.

Thank you!