Lefschetz trace formulas for flows on foliated manifolds

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Glances@Manifolds II

The setting

- M a closed manifold, $\dim M = n$.
- $\mathcal{F}$ a codimension one foliation on M.
- $\phi^t : M \to M, t \in \mathbb{R}$ a foliated flow (i.e., ϕ^t takes each leaf to a leaf).

A Lefschetz number of the flow ϕ:

$$L(\phi) = \sum_{j=0}^{n-1} (-1)^j \mathrm{Tr}\,(\phi^* : H^j \to H^j)$$

H^j is some cohomology theory associated to $\mathcal{F}$, Tr is some trace.

The corresponding Lefschetz trace formula:

$$L(\phi) = \text{a contribution of closed orbits and fixed points of the flow.}$$

Simple flows

Definition

A closed orbit c of period l (not necessarily minimal) of the flow ϕ is called simple, if

$$\det(\mathrm{id} - \phi_*^l : T_x\mathcal{F} \to T_x\mathcal{F}) \neq 0, \quad x \in c.$$

Definition

A fixed point x of the flow ϕ is called simple if

$$\det(\mathrm{id} - \phi_*^t : T_xM \to T_xM) \neq 0, \quad t \neq 0.$$

- $\mathrm{Fix}(\phi)$ the fixed point set of ϕ (closed in M).
- M^0 the $\mathcal{F}$-saturation of $\mathrm{Fix}(\phi)$ (the union of leaves with fixed points).
  Observe that M^0 is ϕ-invariant.
- $M^1 = M \setminus M^0$ the transitive point set.

Simple flows

Definition

The foliated flow ϕ is simple, i.e.:

- all of its fixed points and closed orbits are simple,
- its orbits in M^1 are transverse to the leaves:

$$T_x M = \mathbb{R}\, Z(x) \oplus T_x\mathcal{F}, \quad x \in M^1,$$

where Z is the infinitesimal generator of ϕ (a vector field on M).

$\mathcal{F}$ is a foliation almost without holonomy:

If ϕ is simple, then:

- M^0 is a finite union of compact leaves,
- only the leaves in M^0 may have non-trivial holonomy groups.

Guiilemin-Sternberg formula

There is a canonical expression for the right-hand side of the Lefschetz trace formula, which follows from the Guiilemin-Sternberg formula.

In $\mathcal{D}'(\mathbb{R}^+)$,

$$L(\phi) = \sum_c l(c) \sum_{k=1}^{\infty} \varepsilon_{kl(c)}(c)\delta_{kl(c)} + \sum_p \varepsilon_p |1 - e^{\varkappa_p t}|^{-1},$$

c runs over all closed orbits and p over all fixed points of ϕ:

- $l(c)$ the minimal period of c,
- $\varepsilon_l(c) := \text{sign}\det\left(\text{id} - \phi^l_* : T_x\mathcal{F} \to T_x\mathcal{F}\right), x \in c.$
- $\varepsilon_p := \text{sign}\det\left(\text{id} - \phi^t_* : T_p\mathcal{F} \to T_p\mathcal{F}\right), t > 0.$
- $\varkappa_p \neq 0$ is a real number such that

$$\bar{\phi}^t_* : T_pM/T_p\mathcal{F} \to T_pM/T_p\mathcal{F}, \quad x \mapsto e^{\varkappa_p t}x.$$

Problems

Problem

To define a Lefschetz number of the flow ϕ:

$$L(\phi) = \sum_{j=0}^{n-1} (-1)^j \mathrm{Tr}\,(\phi^* : H^j \to H^j)$$

- H^j is some cohomology theory associated with $\mathcal{F}$,
- Tr is a trace,

in such a way that the above Guillemin-Sternberg formula holds.

Motivation:

Deninger's program to study zeta- and L-functions for algebraic schemes over the integers, in particular, the Riemann zeta-function (Berlin, ICM, 1998).

Nonsingular flows

ASSUMPTIONS:

- M a closed manifold, $\dim M = n$.
- $\mathcal{F}$ a codimension one foliation on M.
- $\phi^t : M \to M, t \in \mathbb{R}$ a simple foliated flow.
- ϕ has no fixed points.

Jesús A. Álvarez López, Y. K., Distributional Betti numbers of transitive foliations of codimension one. Foliations: geometry and dynamics (Warsaw, 2000), 159–183, World Sci. Publ., River Edge, NJ, 2002.

Leafwise de Rham complex

$(\Omega(\mathcal{F}), d_{\mathcal{F}})$ the leafwise de Rham complex of $\mathcal{F}$:

- $\Omega^{\cdot}(\mathcal{F}) = C^{\infty}(M, \Lambda^{\cdot} T^{*}\mathcal{F})$ smooth leafwise differential forms;
- $d_{\mathcal{F}} : \Omega^{\cdot}(\mathcal{F}) \to \Omega^{\cdot+1}(\mathcal{F})$ the leafwise de Rham differential.

In a foliated chart with coordinates $(x_1, \dots, x_{n-1}, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ such that leaves are given by $y = c$, a p-form $\omega \in \Omega^{p}(\mathcal{F})$ is written as

$$\omega = \sum_{\alpha_1 < \alpha_2 < \ldots < \alpha_p} a_{\alpha}(x, y) dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_p}$$

and $d_{\mathcal{F}}\omega \in \Omega^{p+1}(\mathcal{F})$ is given by

$$d_{\mathcal{F}}\omega = \sum_{j=1}^{n-1} \sum_{\alpha_1 < \alpha_2 < \ldots < \alpha_p} \frac{\partial a_{\alpha}}{\partial x_j}(x, y) dx_j \wedge dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_p}$$

Leafwise de Rham cohomology

- The reduced leafwise de Rham cohomology of $\mathcal{F}$:

$$\overline{H}(\mathcal{F}) = \ker d_{\mathcal{F}} / \overline{\operatorname{im} d_{\mathcal{F}}},$$

  the closure is in C^∞-topology.
- ϕ is a foliated flow $\Longrightarrow d_{\mathcal{F}} \circ \phi^t = \phi^t \circ d_{\mathcal{F}}$.
  The induced action:

$$\phi^{t*} : \overline{H}(\mathcal{F}) \to \overline{H}(\mathcal{F}).$$

Question

The trace of $\phi^{t*} : \overline{H}(\mathcal{F}) \to \overline{H}(\mathcal{F})$?

The leafwise Hodge decomposition

- $\mathcal{F}$ is a Riemannian foliation.
- g the Riemannian metric on M such that the infinitesimal generator Z of the flow ϕ is of length one and is orthogonal to the leaves — a bundle-like metric.
- $\Delta_{\mathcal{F}} = d_{\mathcal{F}}\delta_{\mathcal{F}} + \delta_{\mathcal{F}}d_{\mathcal{F}}$ the leafwise Laplacian on $\Omega(\mathcal{F})$ (a second order tangentially elliptic differential operator on M).
- $\mathcal{H}(\mathcal{F})$ the space of leafwise harmonic forms on M:

$$\mathcal{H}(\mathcal{F}) = \{\omega \in \Omega(\mathcal{F}) : \Delta_{\mathcal{F}}\omega = 0\}.$$

Theorem (Alvarez Lopez - Yu. K)

The Hodge isomorphism

$$\overline{H}(\mathcal{F}) \cong \mathcal{H}(\mathcal{F}).$$

Transverse ellipticity

The leafwise de Rham complex $(\Omega(\mathcal{F}), d_{\mathcal{F}})$ of $\mathcal{F}$ as well as the leafwise Laplacian $\Delta_{\mathcal{F}}$ are transversally elliptic relative to the action of the group $\mathbb{R}$, given by the flow ϕ

- The principal symbol $\sigma(\Delta_{\mathcal{F}})$ of $\Delta_{\mathcal{F}}$ is a section of the vector bundle $\mathrm{Hom}(\pi^*\Lambda^\cdot T^*\mathcal{F})$ over T^*M (actually, a scalar function). Here $\pi : T^*M \to M$ is a natural projection.
- Transverse ellipticity means that $\sigma(\Delta_{\mathcal{F}})(x,\xi)$ is invertible for any $(x,\xi) \in T^*M \setminus 0$ such that $\langle \xi, Z(x)\rangle = 0$, where Z is the infinitesimal generator of ϕ
(in other words, for any $(x,\xi) \in T^*M$ orthogonal to the orbits of ϕ).

The index theory for transversally elliptic operators

- Atiyah-Singer, 1973-1974, for compact Lie groups;
- Singer-Hörmander, 1974, definition of the index for noncompact Lie groups.

For any $f \in C_c^\infty(\mathbb{R})$, define

$$A_f = \int_{\mathbb{R}} \phi^{t*} \cdot f(t)\, dt \circ \Pi : L^2\Omega(\mathcal{F}) \to L^2\Omega(\mathcal{F}),$$

where $\Pi : L^2\Omega(\mathcal{F}) \to L^2\mathcal{H}(\mathcal{F})$ is the orthogonal projection.

A_f is a smoothing operator:

For any $f \in C_c^\infty(\mathbb{R})$, the Schwartz kernel $K_{A_f} = K_{A_f}(x,y)|dy|$ is smooth:

$$A_f u(x) = \int_M K_{A_f}(x,y) u(y) |dy|.$$

The Lefschetz distribution

In particular, A_f is of trace class and

$$\operatorname{Tr} A_f = \int_M \operatorname{tr} K_{A_f}(x,x)|dx|.$$

The Lefschetz distribution $L(\phi) \in \mathcal{D}'(\mathbb{R})$:

$$< L(\phi), f >= \operatorname{Tr}^s A_f := \sum_{j=1}^{n-1} (-1)^j \operatorname{Tr} A_f^{(i)}, \quad f \in C_c^\infty(\mathbb{R}),$$

where $A_f^{(i)}$ is the restriction of A_f to $\Omega^i(\mathcal{F})$.

The Lefschetz formula

Theorem (Alvarez Lopez - Y.K.)

Assume that ϕ is simple and has no fixed points.

- *On $\mathbb{R} \setminus \{0\}$*
$$L(\phi) = \sum_c l(c) \sum_{k \neq 0} \varepsilon_{kl(c)}(c) \delta_{kl(c)},$$
when c runs over all closed orbits of ϕ and $l(c)$ denotes the minimal period of c.
- *In some neighborhood of 0 in $\mathbb{R}$:*
$$L(\phi) = \chi_\Lambda(\mathcal{F}) \cdot \delta_0.$$

$\chi_\Lambda(\mathcal{F})$ the Λ-Euler characteristic of $\mathcal{F}$ given by the holonomy invariant transverse measure Λ (Connes, 1979).

The setting

ASSUMPTION:

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- $\phi^t : M \to M, t \in \mathbb{R}$ a simple foliated flow.

- Fix(ϕ) the fixed point set of ϕ (closed in M).
- M^0 the $\mathcal{F}$-saturation of Fix(ϕ) (the union of leaves with fixed points).
- $M^1 = M \setminus M^0$ the transitive point set.

Definition

The foliated flow ϕ is simple, i.e.:

- all of its fixed points and closed orbits are simple,
- its orbits in M^1 are transverse to the leaves.

Difficulties

- The leafwise de Rham complex $(\Omega(\mathcal{F}), d_{\mathcal{F}})$ of $\mathcal{F}$ as well as the leafwise Laplacian $\Delta_{\mathcal{F}}$ are transversally elliptic only on the transitive point set M^1, not on M^0.
- As a consequence, the operator

$$A_f = \int_{\mathbb{R}} \phi^{t*} \cdot f(t)\, dt \circ \Pi : L^2\Omega(\mathcal{F}) \to L^2\Omega(\mathcal{F})$$

  is not a smoothing operator. Its Schwartz kernel is smooth on $M^1 \times M^1$ and singular near $M^0 \times M^0$.
  So its trace is not well-defined.
- I don't discuss the leafwise de Rham complex $(\Omega(\mathcal{F}), d_{\mathcal{F}})$ and related Hodge theory (an open problem).
  Observe that $\mathcal{F}$ is not Riemannian.

The transitive point set and its blow-up

- M^1_l, $l = 1, \ldots, r$, the connected components of M^1:

$$(M^1, \mathcal{F}^1) = \bigsqcup_l (M^1_l, \mathcal{F}^1_l).$$

- M^l is the closure of M^1_l: $M^l = \overline{M^1_l}$.
  Thus, M_l is a connected compact manifold with boundary, endowed with a smooth foliation $\mathcal{F}_l$ tangent to the boundary.
- Put $M^{\mathrm{c}} := \bigsqcup_l M_l, \mathcal{F}^{\mathrm{c}} := \bigsqcup_l \mathcal{F}_l$.

A smooth surjective local embedding $\pi : (M^{\mathrm{c}}, \mathcal{F}^{\mathrm{c}}) \to (M, \mathcal{F})$:

- for each l, $\pi : \mathring{M}_l \to M^1_l$ is a diffeomorphism;
- $\pi : \partial M^{\mathrm{c}} \to M^0$ is a 2-fold covering map whose restriction to every connected component is a diffeomorphism.

ϕ^t lifts to a simple foliated flow $\phi^{\mathrm{c},t}$ of $\mathcal{F}^{\mathrm{c}}$ tangent to ∂M^{c}.

Riemannian metric of bounded geometry

There is a Riemannian metric g^1 on the transitive point set M^1:

- g^1 is bundle-like for $\mathcal{F}^1$;
- M^1_l equipped with $g_l := g^1|_{M^1_l}$ is a manifold of bounded geometry;
- $\mathcal{F}^1_l$ a Riemannian foliation of bounded geometry;
- ϕ^t_l a flow of bounded geometry.

Remarks:

- Observe that g^1 is singular at M^0.
- Each (M^1_l, g^1_l) is a Riemannian manifold with cylindrical ends.

We use a very concrete choice of such a metric g^1. To introduce it, we need to describe a local structure of the foliation near M^0.

Suspension construction

Using the local stability theorem, one can show that $\mathcal{F}$ can be described around a leaf L in M^0 by using the suspension construction.

The data:

- L a connected closed manifold;
- a homomorphism (the holonomy homomorphism)

$$\bar{h} : \Gamma := \pi_1 L / \ker h \to \mathrm{Diffeo}_+(\mathbb{R}, 0), \gamma \mapsto \bar{h}_\gamma, \quad \bar{h}_\gamma(x) = a_\gamma x,$$

where $\gamma \in \Gamma \mapsto a_\gamma \in \mathbb{R}^+$ is a homomorphism.

Suspension construction

The holonomy covering

$\pi : \widetilde{L} \to L$ the regular covering map with

$$\pi_1 \widetilde{L} \equiv \ker h \Leftrightarrow \mathrm{Aut}(\pi) \equiv \Gamma.$$

The canonical left action of each $\gamma \in \Gamma$ on $\widetilde{L}$ is denoted by $\tilde{y} \mapsto \gamma \cdot \tilde{y}$.

The suspension manifold defined by the above data:

$M_L = \widetilde{L} \times_\Gamma \mathbb{R}$ the orbit space for the diagonal Γ-action on $\widetilde{M}_L = \widetilde{L} \times \mathbb{R}$:

$$\gamma \cdot (\tilde{y}, x) = (\gamma \cdot \tilde{y}, a_\gamma x). \quad (\tilde{y}, x) \in \widetilde{L} \times \mathbb{R}.$$

Let $[\tilde{y}, x]$ denote the element in M_L represented by each $(\tilde{y}, x) \in \widetilde{M}_L$.

Suspension construction

The fiber bundle map:

$\widetilde{\varpi} : \widetilde{M}_L = \widetilde{L} \times \mathbb{R} \to \widetilde{L}$ the Γ-equivariant map given by the first factor projection induces the map:

$$\varpi : M_L = \widetilde{L} \times_\Gamma \mathbb{R} \to L, \quad \varpi([\tilde{y}, x]) = \pi(\tilde{y}).$$

Note that the typical fiber of ϖ is $\mathbb{R}$.

The suspension foliation:

$\mathcal{F}_L$ the foliation on M_L transverse to the fibers of $\varpi : M_L \to L$, which is induced by the Γ-invariant foliation on $\widetilde{M}_L$ with leaves $\widetilde{L} \times \{x\}$ $(x \in \mathbb{R})$.

Since 0 is fixed by the Γ-action on $\mathbb{R}$, the leaf $\widetilde{L} \equiv \widetilde{L} \times \{0\}$ of $\widetilde{\mathcal{F}}_L$ projects to a leaf of $\mathcal{F}_L$ that can be canonically identified with L.

Local picture near the compact leaf

According to the local stability theorem, $(M, \mathcal{F})$ is described around L using the suspension foliated manifold $(M_L, \mathcal{F}_L)$.

There are tubular neighborhoods $\varpi : V_L \to L$ of L in M_L and $\varpi : V \to L$ of L in M and a diffeomorphism from V to V_L, which takes $\mathcal{F}|_V$ to $\mathcal{F}_L|_{V_L}$:

$$V \equiv V_L, \quad \mathcal{F}|_V \equiv \mathcal{F}_L|_{V_L}.$$

and the flow ϕ^t on $V \equiv V_L$ is given by

$$\phi^t([\tilde{y}, x]) = [\phi^t_x(\tilde{y}), e^{\varkappa_L t} x], \quad [\tilde{y}, x] \in V_L \subset M_L = \widetilde{L} \times_\Gamma \mathbb{R}.$$

Recall that $\varkappa_p \neq 0$ is a real number (depending only on L) such that

$$\bar{\phi}^t_* : T_pM/T_p\mathcal{F} \to T_pM/T_p\mathcal{F}, \quad x \mapsto e^{\varkappa_p t} x.$$

Riemannian metrics

- g^0 a Riemannian metric on L.
- $g_{\mathcal{F}_L}$ a leafwise Riemanian metric on $(M_L, \mathcal{F}_L)$, defined by requiring that the restrictions of the map

$$\varpi : M_L = \widetilde{L} \times_\Gamma \mathbb{R} \to L, \quad \varpi([\tilde{y}, x]) = \pi(\tilde{y}),$$

  to the leaves of $\mathcal{F}_L$ are local isometries.
- g_{M_L} a Riemannian metric on $M_L \setminus L = \widetilde{L} \times_\Gamma (\mathbb{R} \setminus \{0\})$:

$$g_{M_L} = g_{\mathcal{F}_L} + \frac{dx^2}{x^2}, \quad [\tilde{y}, x] \in \widetilde{L} \times_\Gamma (\mathbb{R} \setminus \{0\}),$$

  is bundle-like for $\mathcal{F}_L$.

Riemannian metrics

We fix an identification

$$V \equiv V_L, \quad \mathcal{F}|_V \equiv \mathcal{F}_L|_{V_L},$$

and easily get a bundle-like metric g^1 on $(M^1, \mathcal{F}^1)$ with the above properties:

- g^1 is bundle-like for $\mathcal{F}^1$;
- M_l equipped with $g_l := g^1|_{M^1_l}$ is a manifold of bounded geometry;
- $\mathcal{F}^1_l$ a Riemannian foliation of bounded geometry;
- ϕ^t_l a flow of bounded geometry.

Operators on the transitive point set

- $M^{c} = \bigsqcup_l M_l, \mathcal{F}^{c} = \bigsqcup_l \mathcal{F}_l$, where M_l is a connected compact manifold with boundary, endowed with a smooth foliation $\mathcal{F}_l$ tangent to the boundary.
- $d_{\mathring{\mathcal{F}}_l}$ the leafwise de Rham differential on $\Omega(\mathring{\mathcal{F}}_l)$.
- $\delta_{\mathring{\mathcal{F}}_l}$ the leafwise de Rham codifferential on $\Omega(\mathring{\mathcal{F}}_l)$.
- $D_{\mathring{\mathcal{F}}_l} = d_{\mathring{\mathcal{F}}_l} + \delta_{\mathring{\mathcal{F}}_l}$.

For any function $\psi : \mathbb{R} \to \mathbb{C}$ of class $\mathcal{A}$, $f \in C^{\infty}_{c}(\mathbb{R})$ and index l, the operator

$$\mathring{P}_l = \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\,dt \circ \psi(D_{\mathring{\mathcal{F}}_l})$$

is a smoothing operator on $\mathring{M}_l$, but its kernel is singular near $\partial \mathring{M}_l$.

Class $\mathcal{A}$

Definition (Roe1987)

Let $\mathcal{A}$ be the Fréchet algebra of functions $\psi : \mathbb{R} \to \mathbb{C}$ that can be extended to entire functions on $\mathbb{C}$ such that, for each compact subset K of $\mathbb{R}$, the set $\{ x \mapsto \psi(x + iy) \mid y \in K \}$ is bounded in the Schwartz space $\mathcal{S}(\mathbb{R})$.

$\mathcal{A}$ contains all functions with compactly supported Fourier transform, as well as the Gaussians $x \mapsto e^{-tx^2}$ with $t > 0$.
By the Paley-Wiener theorem, the Fourier transform $\hat{\psi}$ of any $\psi \in \mathcal{A}$ satisfies that, for every $k \in \mathbb{N}$, there is some $A_k > 0$ such that, for all $\xi \in \mathbb{R}$,

$$|\hat{\psi}(\xi)| \leq A_k e^{-k|\xi|} .$$

b-calculus (R. Melrose)

Theorem (Alvarez Lopez, Yu.K., Leichtnam)

$\mathring{P}_l = \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\,dt \circ \psi(D_{\mathring{\mathcal{F}}_l})$ *gives rise to* $P_l \in \Psi_{\mathrm{b}}^{-\infty}(M_l; \bigwedge T\mathcal{F}_l^*)$.

- The Schwartz kernel K_{P_l} is smooth in the interior $\mathring{M}_l \times \mathring{M}_l$.
- K_{P_l} has a C^∞ extension to $M_l \times M_l \setminus \partial M_l \times \partial M_l$ that vanishes to all orders at $(\partial M_l \times M_l) \cup (M_l \times \partial M_l)$.
- Consider a tubular neighborhood of $L \subset \pi_0(\partial M_l)$ with coordinates $(\rho, y), \rho \in (0,\infty), y \in L$.
  Then $K_{P_l} = K_{P_l}(\rho, y, \rho', y')u(\rho', y')|d\rho'||dy'|$ has the form

$$K_{P_l}(\rho, y, \rho', y') = \frac{1}{\rho'}\kappa_{P_l}(\rho, y, \frac{\rho'}{\rho}, y'),$$

  where $\kappa_{P_l}(\rho, y, s, y')$ is smooth up to L (that is, up to $\rho = 0$).

b-trace

In a tubular neighborhood of L with coordinates $\rho \in (0, \epsilon_0), y \in L$,

$$P_l u(\rho, y) = \int K_{P_l}(\rho, y, \rho', y') u(\rho', y') |d\rho'||dy'|,$$

$$K_{P_l}(\rho, y, \rho', y') = \frac{1}{\rho'} \kappa_{P_l}(\rho, y, \frac{\rho'}{\rho}, y'),$$

and $\kappa_{P_l}(\rho, y, s, y')$ is smooth up to L (that is, up to $\rho = 0$).

Definition

$$ {}^{\mathrm{b}}\mathrm{Tr}\,(P_l) = \lim_{\epsilon \to 0} \left(\int_{\rho > \epsilon} K_{P_l}(\rho, y, \rho, y) |d\rho||dy| + \ln \epsilon \int \kappa_{P_l}(0, y, 1, y) |dy| \right).$$

Key fact

The functional ${}^{\mathrm{b}}\mathrm{Tr}$ doesn't have trace propertry, but ${}^{\mathrm{b}}\mathrm{Tr}\,[P, P']$ is expressed in terms of traces of some explicit integral operators on ∂M_l.

Operators on the transitive point set

Since $M^{\mathrm{c}} = \bigsqcup_l M_l, \mathcal{F}^{\mathrm{c}} = \bigsqcup_l \mathcal{F}_l$, we get the operator

$$P \equiv \bigoplus_l P_l = \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\, dt \circ \psi(D_{\mathcal{F}^{\mathrm{c}}})$$

$$\in \Psi_{\mathrm{b}}^{-\infty}(M^{\mathrm{c}}; \bigwedge T\mathcal{F}^{\mathrm{c}*}) \equiv \bigoplus_l \Psi_{\mathrm{b}}^{-\infty}(M_l; \bigwedge T\mathcal{F}_l^*)\ .$$

In particular, its b-trace ${}^{\mathrm{b}}\mathrm{Tr}\,(P)$ is well-defined.
The b-supertrace of P:

$${}^{\mathrm{b}}\mathrm{Tr}\,^{\mathrm{s}}(P) = \sum_{j=1}^{n-1} (-1)^j\, {}^{\mathrm{b}}\mathrm{Tr}\,(P^{(j)}),$$

where $P^{(j)}$ is the restriction to j-forms.

Heat equation approach to the index theorem

- For $u > 0$, $\psi_u(x) = e^{-u^2x^2}$, $x \in \mathbb{R}$, and, accordingly,

$$P_u = \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\, dt \circ e^{-u^2 D^2_{\mathcal{F}^c}}.$$

- $\frac{d}{du} \operatorname{Tr}^s P_u = 0$, which means that $\operatorname{Tr}^s P_u$ is independent of u.
- As $u \to +\infty$,

$$\operatorname{Tr}^s P_u \to \operatorname{Tr}^s \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\, dt \circ \Pi = \langle \mathcal{L}(\phi), f \rangle.$$

- As $u \to 0$, $\operatorname{Tr}^s P_u$ can be computed, using heat kernel approximations (fantastic cancellations).

Remark

In our case, b-trace ${}^{\mathrm{b}}\mathrm{Tr}$ *doesn't satisfy the trace property. Therefore,*

$$\frac{d}{du} {}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}} P_u \neq 0.$$

Some geometric notions

- There is some real number $\varkappa_L \neq 0$ such that, for $p \in L$,

$$\bar{\phi}^t_* : N_p\mathcal{F} \to N_p\mathcal{F}, \quad x \to e^{\varkappa_L t} x.$$

- The holonomy homomorphism

$$\bar{h}_L : \Gamma := \pi_1 \tilde{L} \to \mathrm{Diffeo}_+(\mathbb{R}, 0), \quad \gamma \mapsto \bar{h}_{L,\gamma},$$

  with $\bar{h}_{L,\gamma}(x) = a_{L,\gamma} x$ for some homomorphism $\Gamma \to \mathbb{R}^+$, $\gamma \mapsto a_{L,\gamma}$.
- Relative periods:

$$t_{L,\gamma} = -\varkappa_L^{-1} \log a_{L,\gamma} \;.$$

Some geometric notions

- Fix generators $\gamma_1, \dots, \gamma_k$ of Γ ($k = \operatorname{rank} \Gamma$).
- c_i a piecewise smooth loop in L based at p representing γ_i^{-1}.
- Using the universal coefficients and Hurewicz theorems, one can show that there are closed 1-forms $\beta_1, \dots, \beta_k$ on L such that $\delta_{ij} = \langle [\beta_i], \gamma_j \rangle = -\int_{c_j} \beta_i$, and $\langle [\beta_i], \ker h \rangle = 0$.
- Consider a closed 1-form η on L:

$$\eta = \ln(a_{\gamma_1})\, \beta_1 + \dots + \ln(a_{\gamma_k})\, \beta_k \ ,$$

  and $\tilde{\eta}$ is the lift of η to $\widetilde{L}$.
- If we consider η as a closed leafwise 1-form on the suspension manifold M_L, then there exists a 1-form ω on M_L satisfying $T\mathcal{F}_L = \ker \omega$ such that

$$d\omega = \eta \wedge \omega \ .$$

Derivative of the b-supertrace

Fix an even $\psi \in \mathcal{A}$ and $f \in C^\infty_c(\mathbb{R})$.
For $u > 0$, let $\psi_u \in \mathcal{A}$, $\psi_u(x) = \psi(ux)$ and

$$P_{\psi_u,f} = \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\, dt \circ \psi_u(D_{\mathcal{F}^c})$$

Theorem

$$\frac{d}{du}{}^{\mathrm{b}}\mathrm{Tr}^{\,s}(P_{\psi_u,f}) = \sum_{L\in\pi_0(M^0)} \frac{2}{|\varkappa_L|} \sum_{\gamma\in\Gamma_L} \mathrm{Tr}^{\,s}_{\Gamma_L}\left(T^*_\gamma \widetilde{R}_{\widetilde{L},u,t_{L,\gamma}}\right) f(t_{L,\gamma}) ,$$

where Tr_{Γ_L} is the Γ_L-trace of Γ_L-invariant operators on $\widetilde{L}$ and

$$\widetilde{R}_{\widetilde{L},u,t} = \tilde{\eta} \wedge \tilde{\phi}^{t*}_0 \psi'_u(D_{\widetilde{L}})$$

Variation of the b-supertrace and Lefschetz distribution

For $u, v > 0$,

$$
{}^{b}\mathrm{Tr}^{\,s}(P_{\psi_v,f}) - {}^{b}\mathrm{Tr}^{\,s}(P_{\psi_u,f}) = \sum_{L\in\pi_0(M^0)} \frac{2}{|\varkappa_L|} \sum_{\gamma\in\Gamma_L} \mathrm{Tr}^{\,s}_{\Gamma_L}\left(T^*_\gamma \widetilde{S}_{\widetilde{L},u,v,t_{L,\gamma}}\right) f(t_{L,\gamma}) ,
$$

$$
\widetilde{S}_{\widetilde{L},u,v,t} = \int_u^v \widetilde{R}_{\widetilde{L},w,t}\, dw = \widetilde{\eta}\wedge \tilde{\phi}_0^{t*} \frac{\psi(vD_{\widetilde{L}}) - \psi(uD_{\widetilde{L}})}{D_{\widetilde{L}}} .
$$

The Lefschetz distribution

$$
\langle L(\phi), f\rangle = {}^{b}\mathrm{Tr}^{\,s}(P_{\psi_v,f}) - \lim_{u\to 0} \sum_{L\in\pi_0(M^0)} \frac{2}{|\varkappa_L|} \sum_{\gamma\in\Gamma_L} \mathrm{Tr}^{\,s}_{\Gamma_L}(T^*_\gamma \widetilde{S}_{\widetilde{L},u,v,t_{L,\gamma}})\, f(t_{L,\gamma}).
$$

Here the right-hand side is independent of v.

Lefschetz distribution

Theorem

There exists the limit of ${}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{\psi_u,f})$ *as* $u \to 0$*, which is given on* $\mathbb{R}_+$ *by*

$$\lim_{u\to 0} {}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{\psi_u,f}) = \sum_c l(c) \sum_{k=1}^{\infty} \varepsilon_{kl(c)}(c) \cdot f(kl(c))$$

where c *runs over all closed orbits of* ϕ^t*,* $l(c)$ *denotes the minimal period of* c*, and* x *is an arbitrary point of* c*.*

Trace formula

Corollary

$L(\phi)$ is a well-defined distribution on $\mathbb{R}_+$ and

$$\langle L(\phi), f\rangle = \lim_{u\to 0} {}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{\psi_u,f}).$$

Theorem

We have

$$L(\phi) = \sum_c l(c) \sum_{k=1}^{\infty} \varepsilon_{kl(c)}(c)\cdot \delta_{kl(c)}$$

on $\mathbb{R}_+$, where c runs over all closed orbits of ϕ^t, $l(c)$ denotes the minimal period of c, and x is an arbitrary point of c.

Concluding remarks

Remark

The next problem is to give a cohomological interpretation of the limit as $v \to +\infty$ of

$$ {}^{b}\mathrm{Tr}^{s}(P_{\psi_v,f}) - \lim_{u\to 0} \sum_{L\in\pi_0(M^0)} \frac{2}{|\varkappa_L|} \sum_{\gamma\in\Gamma_L} \mathrm{Tr}^{s}_{\Gamma_L}\left(T^*_\gamma \widetilde{S}_{\widetilde{L},u,v,t_{L,\gamma}}\right) f(t_{L,\gamma}). $$

Remark

Contribution of fixed points as in the Guillemin-Sternberg formula.