The coarse Baum-Connjecture for product of nonpositive curved spaces and groups

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This talk is based on
the joint work with OGUNI Shin-ichi （尾國新一）

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Coarse Baum-Connes conjecture (Roe, Higson, Yu, . . .)

- Y : proper metric space
- $KX_\bullet(Y)$: coarse K-homology of Y
- $C^*(Y)$: a C^*-algebra constructed by Y, called Roe algebra, which is a non-equivariant analog of the reduced group C^*-algebra.

Conjecture (coarse Baum-Connes)

The following coarse assembly map is an isomorphism.

$$\mu_Y : KX_\bullet(Y) \to K_\bullet(C^*(Y)).$$

Proposition

Consider a finitely generated group G.
Suppose that BG is realized by a finite simplicial complex.

- μ_G *: isomorphism* $\Rightarrow$ *Novikov conj. for G holds.*
- $\mu_G \otimes 1_{\mathbb{Q}}$ *: injective* $\Rightarrow$ *Gromov-Lawson conj. for G holds.*

Advantage of the coarse geometry

coarse assembly map $\mu_Y \colon KX_\bullet(Y) \to K_\bullet(C^*(Y))$.

- $KX_\bullet(-)$, $K_\bullet(C^*(-))$: Coarse Homology Theory
- Mayer-Vietoris Principal :
  Decompose $Y = A \cup B$
  If $\mu_A, \mu_B, \mu_{A\cap B}$ are isomorphisms.
  $\Rightarrow$ So is μ_Y.

  - $\mathbb{R}^n = \mathbb{R}^{n-1} \times (-\infty, 0] \cup \mathbb{R}^{n-1} \times [0, \infty)$.
    The intersection $= \mathbb{R}^{n-1} \times \{0\}$.

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Theorem (Higson-Roe, Willet)

The coarse Baum-Connes conjecture holds for geodesic Gromov hyperbolic spaces.

Theorem (Higson-Roe, Willet, O-F)

The coarse Baum-Connes conjecture holds for CAT(0)-spaces (more generally, for Busemann non-positive curved spaces)

(REMARK: Above theorems hold without assuming bounded geometry condition)

Theorem (Yu)

If Y can be coarsely embedded into the Hilbert space $\Rightarrow$ the coarse Baum-Connes conjecture holds for Y.

Remark

- It is unknown that Gromov hyperbolic space without bounded geometry can be embedded into the Hilbert space.
- It is unknown that all CAT-(0) spaces can be embedded in to the Hilbert space.

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Summary of our results

with Shin-ichi OGUNI, we obtain the following results:

- The conjecture holds for the relatively hyperbolic groups with some conditions on parabolic subgroups (2012).
- Moreover, the conjecture holds for the direct product of hyperbolic groups, CAT(0)-groups, polycyclic groups and relatively hyperbolic groups with some conditions on parabolic subgroups (2015).

We also constructed a nice boundary of the relatively hyperbolic group, and

- Compute the K-theory of certain C^*-algebra.
- Prove the formula to determine the topological dimension of the boundary by the cohomological dimension of the group.

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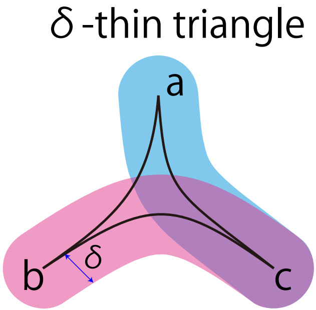
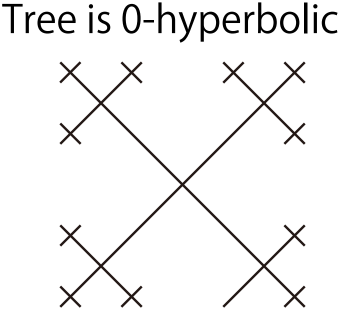
δ-hyperbolic space

Definition

Let $\delta \geqslant 0$. A proper geodesic space X is δ-hyperbolic if all geodesic triangle are δ-thin, i.e., for any $a, b, c \in X$, $\overline{ab}$ is contained in the δ-neighborhood of $\overline{bc} \cup \overline{ca}$.





Hyperbolic groups

Let G be a finitely generated group.

Definition

G is hyperbolic if the following conditions are satisfied.

- ${}^{\exists}X$: a proper geodesic δ-hyperbolic space,
- $G \circlearrowright X$ properly discontinuously by isometries,
- X/G is compact.

Remark

G is hyperbolic $\Leftrightarrow \mathrm{Cayley}(\mathrm{G}, \mathrm{S})$ *δ-hyperbolic for some* $\delta \geqslant 0$ *and for some generating set* S.

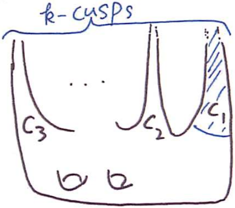
Examples of hyperbolic groups

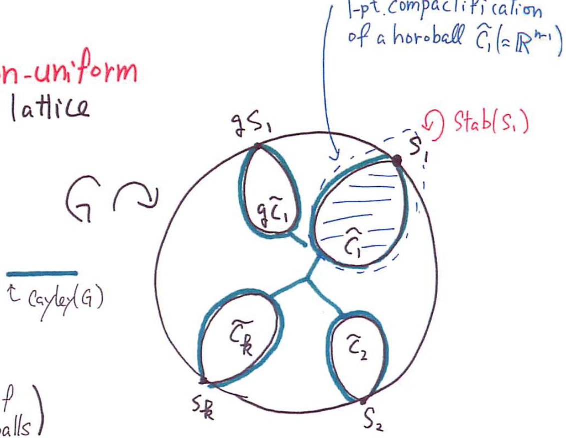
- Free group $F_2 = \langle a, b \rangle$
- $\pi_1(M_g)$, where M_g is a closed surface of genus $g \geqslant 2$.
- Let $G < \mathrm{Isom}(\mathbb{H}^n)$ be a torsion-free cocompact lattice, i.e. $\mathbb{H}^n/G$ is a compact hyperbolic manifold. Then $G \cong \pi_1(\mathbb{H}/G)$ is a hyperbolic group.
- $\pi_1(M)$: where M is a compact Riemannian manifold with strictly negative sectional curvature.

Non-example: Non-uniform (torsion-free) lattice of $\mathbb{H}^n$.

$G < \mathrm{Isom}(\mathbb{H}^n)$ non-uniform lattice

$M := \mathbb{H}^n/G$





$\overline{\mathbb{H}^n} \supset \mathbb{H}^n \underset{\text{isom}}{\approx} \tilde{M}$

- $Cay(G) \underset{\text{coarsely equiv.}}{\approx} \mathbb{H}^n \setminus \left(\text{union of all horoballs}\right) = \mathbb{H}^n \setminus \left\{ \bigcup_{1 \le i \le k} \bigcup_{g} g\tilde{C}_i \right\}$
- $G > \mathrm{Stab}(s_1) > \mathbb{Z}^{n-1}$

$\Rightarrow$ G is not hyperbolic ($n \geq 3$).

Relatively hyperbolic groups

- G: a finitely generated group.
- $\mathbb{P} := \{P_1, \ldots, P_k\}$: a finite family of infinite subgroups.

There is a rigorous definition (formulation) of
" G is hyperbolic relative to $\mathbb{P}$ "

Example

- Free product
  - $\mathbb{Z}^n * \mathbb{Z}^n$ is hyperbolic rel. to $\{\mathbb{Z}^n, \mathbb{Z}^n\}$.
- Fundamental group of a manifold with negative curvature.
  - M : completed Riemann mfd, non-cpt, finite volume. $n := \dim M$, $-\alpha^2 < K_M < -\beta^2 (\alpha, \beta \in \mathbb{R})$
  - $G := \pi_1(M)$.
  - $\mathbb{P}$: a set of representatives of conjugacy invariant classes of maximal parabolic subgroups of G with respect to the action on the universal cover $\tilde{M}$.
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Main theorem 1

Theorem (Oguni-F '12)

- *G : finitely generated group*
- $\mathbb{P} = \{P_1, \ldots, P_k\}$: $P_i < G$, $\sharp P_i = \infty$, $[G : P_i] = \infty$.
- *G is hyperbolic relative to $\mathbb{P}$.*

For each $P \in \mathbb{P}$, we suppose the following two conditions:

- *The space BP is realized by a finite simplicial complex.*
- *The coarse Baum-Connes conjecture for P holds.*

Then the following two statements holds.

- *The space BG is realized by a finite simplicial complex.*
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Remark

It is unknown that under the above condition, whether G can be embedded into the Hilbert space or not.

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Main Theorem 2

Theorem (Oguni-F '15)

Let $\mathcal{G}$ be a class of groups consists of CAT(0)-groups, hyperbolic groups and polycyclic groups.

- *For $i \in \{1, \ldots, n\}$, let G_i be a group belongs to $\mathcal{G}$.*
- *For $j \in \{1, \ldots, m\}$, let H_j be a relatively hyperbolic group with the condition $\sharp_j$ (See the below)*
- *Set $\mathbb{G} := \prod_{i=1}^{n} G_i \times \prod_{i=1}^{m} H_j$.*

Then the coarse Baum-Connes conjecture holds for $\mathbb{G}$.

Condition ($\sharp_j$.)

H_j is hyperbolic relative to a finite family $\mathbb{P}^j$, and for each $P \in \mathbb{P}^j$,

- $P < H_j$, $[H_j : P] = \infty$, $\sharp P = \infty$,
- *P is a direct product of finite members of $\mathcal{G}$,*
- *the space BP is realized by a finite simplicial complex.*

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Metric spaces version

Theorem (Oguni-F '15)

Let $\{X_i\}_{i=1}^n$ be a finite seq. of proper metric spaces. Suppose that X_i is one of the following:

- *a geodesic Gromov hyperbolic space,*
- *an open cone over a compact metrizable space,*
- *a Busemann non-positively curved space, or ,*
- *simply connected solvable Lie group with a lattice,*

Then the coarse assembly map

$$\mu(X_1 \times \cdots \times X_n)_\bullet$$

is an isomorphism.

Key to the proof: Boundary of the product space

The main ingredient of the proof is to construct a NICE boundary for a product of metric space $X_1 \times \cdots \times X_n$ by the topological join

$$\partial X_1 * \cdots * \partial X_n.$$

Key Proposition

Proposition

Let $\{(X_i, W_i)\}_{i=1}^{n}$ be a finite seq. of the pairs of proper metric space and compact metrizable space.
Suppose that (X_i, W_i) is one of the following:

- $(X_i, \partial X_i)$: *a geodesic Gromov hyperbolic space and the Gromov boundary.*
- $(\mathcal{C}(W_i), W_i)$*: a (metric euclidean) cone over* W_i*.*
- $(X_i, \partial X_i)$*: a Busemann non-positively curved space and the visual boundary, or*
- (G_i, S^{n_i})*:* n_i*-dimensional simply connected solvable Lie group with a lattice, and* n_i*-dimensional sphere.*

Then the transgression map

$$T\colon KX_*(X_1 \times \cdots \times X_n) \to \tilde{K}_{*-1}(W_1 * \cdots * W_n)$$

is an isomorphism.