Glances at Manifolds II - Krakow

Prospectives in Index Theory

Nicolae Teleman
Dipartimento di Scienze Matematiche,
Universita' Politecnica delle Marche, 60131-Ancona, Italia
e-mail: teleman@dipmat.univpm.it

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Contents

Abstract

In this lecture the following topics will be discussed in chronological order.

1. Direct connections
2. Localise periodic cyclic homology, algebraic K-theory.
3. Algebraic T-theory.
4. Define a new completion procedure of semigroups.
5. Define the analytic and topological indices associated with *any* short exact sequence of associtive algebras.
6. Extend the problem of the index formula for exact sequences of associative algebras.
7. Define *Non-commutative Topology.*

1 Atiyah - Singer Index theorem.

[**?**], [**?**] deal with *smooth* manifolds.

Let (D) be an elliptic operator. Then

$$AnalyticIndex(D) = TopologicalIndex(D) \tag{1}$$

$$Index(D) = Ch(D) \cup Todd(D) \cap [M]. \tag{2}$$

Un this lecture we make reference to [**?**].

2 Teleman Index Formula.

N. Teleman [?], [?] proved the index formula on *topological* manifolds.

This formula tells that the index formula is a *topological statement*.

The Teleman formula expresses the indices of an elliptic operator as co-homological *classes*.

We intend to represent them as co-homologic *chains*.

3 Connes - Moscovici Local Index Formula.

Holds for *differential and pseudo-differential* elliptic operators on *smooth manifolds*. One introduces the following formulas.

Suppose A is a differential or pseudo-differential elliptic operator. Let B be a *parametrix* for the operaotor A. Define $S_0 = 1 - BA$ and $S_1 = 1 - AB$. Let $f_0, f_1 \dots f_n$ be smooth functions on M. One defines

$$Index(D) = Tr(f_0\ R(D)\ f_1\ R(D) \dots f_n\ R(D)) \tag{3}$$

where

$$L(D)\ = \begin{pmatrix} S_0 & -(1+S_0B) \\ A & S_1 \end{pmatrix} \tag{4}$$

$$P(D) = L(D)\ e_1\ L(D)^{-1} \tag{5}$$

and

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{6}$$

$$e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$

$$R(D) = P - e_2 = \begin{pmatrix} S_0^2 & S_0(1+S_0)B \\ S_1A & -S_1^2 \end{pmatrix}. \tag{8}$$

The operator $R(D)$ has the folllowing properties

1. it is a compact operator
2. the operator P is an *idempotent*
3. the *index* of the operator P is the *analytical index* of D.

   Clearly

$$Index(D) = TraceR(D) = Trace(S_0^2) - Trace(S_1^2). \tag{9}$$

$R(D)$ is the image of the *simbol* $[\sigma(D)]$ through the *connecting homomorphism* associated to the exact sequence of pseudo-differential operators $\partial : K^1(C^\infty(M)) \longrightarrow K^0(C^\infty(M))$.

The Connes - Moscovici formula [?] is

$$Tr(f_0R(D)f_1R(D)\dots f_nR(D))\ = \tag{10}$$

$$Cont.Ch(\sigma D)Td(M)[f_0f_1\dots f_n]. \tag{11}$$

The Alexander - Spanier $f_0f_1\dots f_n$ cocycle has *three functions* in this formula.

1. The cycle has to be *totally skew symmetric* for the formula to work.
2. It is used to *localise* the procedure.
3. This formula gives the components of the Chern character of the manifold M, once at a time, for each n in the expression (3).

Remark 1 *The classical K^1-group $K^1(C^\infty(M)) = \mathbb{Z}$ is* too small *to allow obtaining* all components of the Chern character *of the manifold.*

4 Connes - Sullivan - Teleman Index Formula.

This work produces a *local expressions* for the index of an elliptic operator on *topological manifolds*; it solves the problem formulated in section §2.

This method still produces the *components of the Chern character* once of a time. To produce *all components* of the Chern character (the analogue of the Thom - Hirzebruch formula on topological manifolds), we have to introduce *new notions*: a *new connection*, called *direct connection* and a new K-theory, called *T-theory.* This will be shown in the remaining part of this lecture; see also Theorem 6, §11.

5 Direct connections. Characteristic classes.

Definition 2 *Let M be a topological space and $\mathcal{U}$ a neighbourhood of the diagonal in $M \times M$.*

A direct connection on M is a function $A : \mathcal{U} \longrightarrow \mathcal{U}$ with the following properties

1. *for any pair of points $x,\ y \in \mathcal{U}$, $A(x,y)$ is a* homeomorphism *from a neighbourhood $U \subset \mathcal{U}$ of x to a neighbourhood $V \subset \mathcal{U}$ of y*
2. *$A(x,x) \ = \ Identity$.*

The function A may be asked to be smooth/continuous.

The definition may be extended to algebraic structures *over* arbitrary spaces.

On smooth manifolds, the notion of direct connection is more general than the notion of linear connection. Any linear connection on the smooth manifold M defines a direct connection.

Direct connections were introduced on smooth manifolds to extract the Pontrjagin classes of smooth manifolds [**?**]. Later they were extended (and changed the name to "direct connections") on arbitrary smooth manifolds to extract characteristic classes of bundles.

On topological manifolds direct connections could be constructed by using a handlebody decomposition on the manifold.

Let Γ be a linear connection in the bundle ξ over M and let g be a Riemannian metric on the smooth manifold. They define a direct connection in the following way. Suppose x, y are close points in M and let λ be a small geodesic connecting the two points. Then the direct connnection $A(y,x)$ is given by the parallel transport of vectors of ξ over the point x

to the point y along the geodesic λ, see Teleman N. [?], Teleman K., [?], [?], [?], [?], Kubarski J., Teleman N. [?].

[?], Teleman K., [?], [?], [?], [?], [?] show that the direct connection A could be used to obtain the characteristic classes on differentiable manifolds.

6 Modified Hochschild homology.

For this topic refer to [?].

For any associative ring $\mathcal{A}$ introduce the operator

$$\tilde{b}_n \ = \sum_{k=1}^{k=n} (-1)^k C_{2n}^k (db)^{k-1} \tag{12}$$

$$\tilde{b} : C_n(\mathcal{A}) \longrightarrow C_{n-1}(\mathcal{A}). \tag{13}$$

The *modified Hochschild homology* $\tilde{b}$ has the following properties

1. it is a boundary
2. the corresponding homology is at least as big as the homology of the Hochschild complex
3. The modified Hochschild boundary *commutes with the Alexander - Spanier boundary*
$$\tilde{b}\, d \ + \ d\, \tilde{b} \ = 0. \tag{14}$$
4. it can be *localised*.

We recall that the *periodic cyclic homology* is the homology of the bi-complex $(b,\ B)$. We may speak about *periodic local cyclic homology* as the homology of the *localised bi-complex* $(\tilde{b}, d)$.

Recall that the *periodic cyclic homology* of the associative algebra $\mathcal{A}$ is the homology the bi-complex (b, B).

The periodic local cyclic homology is going to be the homology in the *non - commutative topology* introduced in section §8.

7 Algebraic T-theory.

For this topic refer to [?].

The classical algebraic K^i-groups, $i = 0, 1$, are defined as follows,

$$K^0(\mathcal{A}) := \text{conjugated classes of idempotents of the matrix algebra of } \mathcal{A} \tag{15}$$

$$K^1(\mathcal{A}) := GL_*(\mathcal{A})/[GL_*(\mathcal{A}), GL_*(\mathcal{A})]; \tag{16}$$

here we have assumed that $\mathcal{A}$ is an unital associative algebra and $GL_*(\mathcal{A})$ are the invertible elements of the matrix algebra over $\mathcal{A}$.

The new groups are denoted T^i. They are defined as follows, see see Teleman [?]

In both groups T^i the addition is given by the direct sum of equivalence classes of *conjugated matrices*. The elements of $u_1 \sim u_2$ are *conjugated* provided there exists an invertible element u such that

$$u_1 = u\, u_2\, u^{-1}. \tag{17}$$

Definition 3

$$T^0(\mathcal{A}) := Grothendieck\ completion\ of\ conjugated\ classes\ of \tag{18}$$
$$idempotentsof\ the\ matrix\ algebra\ of\ \mathcal{A}. \tag{19}$$

To define T^1 we consider

$$T^1(\mathcal{A}) := \frac{1}{2}\, GL_*(\mathcal{A})/\sim . \tag{20}$$

Two conjugated invertible elements u_1, u_2 are $\sim$ equivalent provided there exist invertible elements $\tilde{\xi}_i$

$$\xi_i = \begin{pmatrix} \tilde{\xi}_i & 0 \\ 0 & \tilde{\xi}_i^{-1} \end{pmatrix} \tag{21}$$

such that

$$u_1 + \xi_1 = u_2 + \xi_2. \tag{22}$$

The factor $\frac{1}{2}$ is necessary to insure that $\sim$ passes to the quotient for an exact sequence of associative algebras

$$0 \longrightarrow \Lambda^{'} \longrightarrow \Lambda_1 \oplus \Lambda_2 \longrightarrow \Lambda \longrightarrow 0 \tag{23}$$

there correponds an exact sequence of T^i groups

$$T_1^{loc}(\Lambda^{'}) \stackrel{(i_{1*}, i_{2*})}{\longrightarrow} T_1^{loc}(\Lambda_1) \oplus T_1^{loc}(\Lambda_2) \stackrel{j_{1*} - j_{2*}}{\longrightarrow} T_1^{loc}(\Lambda^{'}) \stackrel{\partial}{\longrightarrow} \tag{24}$$

$$T_0^{loc}(\Lambda^{'}) \otimes \mathbb{Z}[\frac{1}{2}] \stackrel{(i_{1*}, i_{2*})}{\longrightarrow} T_0^{loc}(\Lambda_1) \otimes \mathbb{Z}[\frac{1}{2}] \oplus T_0^{loc}(\Lambda_2) \otimes \mathbb{Z}[\frac{1}{2}] \stackrel{j_{1*} - j_{2*}}{\longrightarrow} T_0^{loc}(\Lambda^{'}) \otimes \mathbb{Z}[\frac{1}{2}] \tag{25}$$

We expect the factor 1/2 to play an important role in the theory.

8 T-completion

T-completion is used in [**?**] without being nominated.

The group T_1 was obtained by completing the semigroup of conjugacy classes of invertible elements by a *new procedure*. We explain it.

The data is: a semigroup K and a subgroup U. The involution I is assumed to be an involution on K. In our case $K = invertible\ elements$, $I = inversion$ and $U = \{A \oplus A^{-1}\}$.

Two elements u_1, u_2 of the subgroup K are called *equivalent* $\sim$ provided there exists two elements $\xi_1, \xi_2 \in U$ such that

$$u_1 \ + \ \xi_1 \ = \ u_2 \ + \ \xi_2. \tag{26}$$

9 Non-commutative Topology.

For any associative ring $\mathcal{A}$ one associates the *non - commutative topology* over $\mathcal{A}$.

The *homology* in the *non - commutative topology* is the $T_i(\mathcal{A})$-theory and the *periodic local cyclic homology.*

In the next section §10 we will define the Chern character on $T_i(\mathcal{A})$.

10 Chern character of idempotents in Non-commutative Topology

.

Theorem 4 *Let $e \in \mathbb{M}_*$ be an idempotent over the ring $\mathcal{A}$.*

Let

$$\Psi_{2n} \ = \ e(de)^n. \tag{27}$$

Then

$$\tilde{b}\ \Psi_{2n} \ = \ n\ d\Psi_{2n}. \tag{28}$$

Therefore, $\{\Psi_{2n}\ \}_{n\in\mathbb{Z}}$ defines a cycle in the homology of the bi - complex $(\tilde{b},\ d)$. By definition, this is the Chern character *of the idempotent e in non-comutative topology.*

Analogously one defines the Chern character of elements in $T_1(\mathcal{A})$.

Remark 5 *The components of the idempotent e may by discontinuous.*

11 The index formula in Non-commutative Topology.

Let

$$0 \longrightarrow \mathcal{J}^{'} \longrightarrow \mathcal{J} \longrightarrow \mathcal{J}^{"} \longrightarrow 0 \tag{29}$$

be an exact sequence of localised associative rings. We assume the last two rings to be unital. The first ring $\mathcal{J}^{'}$ is going to be unitarised.

Let σ be an invertible element belonging to $\mathbb{M}_{ast}(\mathcal{J}^{"})$. We associate the exact sequence in algebraic T_*-theory.

Let $\sigma \in T_1^{loc}(\mathcal{J}^{"})$ be an invertible element.

STEP I: With this element we associate

1. the *topological index* $TopInd^T(\sigma) := \partial[\sigma] \in T_0^{loc}(\mathcal{J}^{'})$ and the
2. *analytical index* $AnaInd(\sigma) \in T_0^{loc}(\mathcal{J}^{'})$
3. by the same formulas defined in section §3. This means that the analytic index is the Chern character of the element $R(\sigma)$, where $R(\sigma)$ is defined by the formula (8).

   These are the *basic* definitions of the topological and analytical indices. All other notions of analytical and topological indices derive from these two.

It is important to notice that these definitions apply to *any* exact sequence of localised associative rings.

The reader will remark that the *Todd class* does not appear in the formula for the topological index.

STEP II: Apply the Chern character to both topological and analytic indices. One obtains the *topological and analytical indices with values in the local cyclic homology.*

At this point the problem is how much regularity is available. This involves the problem of multiplying distributions.

The problem consists of seeing whether the local cyclic homology classes could be pushed up to the *diagonal*. If the manifold is *smooth and the exact sequence consists of pseudo-differential operators*, the corresponding index theorem is the Atiyah - Singer formula [**?**]. For topological manifolds one obtains the Connes - Sullivan - Teleman [**?**] formula.

Theorem 6 *Let* $e \in \mathcal{J}^{''}$ *be an invertible element. Let* $[e] \in T_1(\mathcal{J}^{''})$ *and* $\partial([e]) \in T_0(\mathcal{J}^{'})$*. This is an idempotent.*

The Chern character of $\partial([e])$ *has components in all even dimensional local periodic cyclic homology.*

If the exact sequence (29) is an exact sequence of pseudo-differential/compact operators on the smooth manifold M *and if* e *is the simbol of the signature operator on* M*, then the periodic local cyclic homology of the Chern character of* e *is the Thom-Hirzebruch characteristic class.*

This theorem may be reformulated on arbitrary quasi-conformal manifolds M.

12 Rational Pontrjagin classes of Topological Manifolds.

In this section M is a *topological* manifold. We assume that A is a *continuous direct connection* on M.

Definition 7 *Let* M *is a* topological *manifold and we assume that* A *is a* continuous direct connection *on* M.

Define

$$\tilde{Phi}_{2n}[x_0, x_1, \ldots, x_{2n}] \;= \qquad (30)$$

$$A(x_0, x_1) \otimes A(x_1, x_2) \otimes \cdots \otimes A(x_{2n}, x_0). \qquad (31)$$

Theorem 8 *Let* M *be a topological manifold and* A *is a* continuous direct connection *on* M.

Then

$$b^{'}(\tilde{Phi}_{2n}) \;=\; 0. \qquad (32)$$

Proof. Having an even sum of alternating terms $A(x_0, x_1) \otimes A(x_1, x_2) \otimes \cdots \otimes A(x_{2n-1}, x_0)$,

$$b^{'}(\tilde{Phi}_{2n})[x_0, x_1, \ldots, x_{2n-1}] \; = \tag{33}$$

$$\sum_{i=0}^{i=2n-1} (-1)^i A(x_0, x_1) \otimes A(x_1, x_2) \otimes \cdots \otimes A(x_i, x_i).A(x_i, x_{i+1}) \otimes \cdots \otimes A(x_{2n}, x_0) \; = \tag{34}$$

$$\sum_{i=0}^{i=2n-1} (-1)^i A(x_0, x_1) \otimes A(x_1, x_2) \otimes \cdots \otimes A(x_i, x_{i+1}) \otimes \cdots \otimes A(x_{2n-1}, x_0) \; = 0 \tag{35}$$

the sum is zero. ■

Theorem 9 *The chain Phi_{2n} has the following properties*

1. *it has* cyclic symmetry
2. *it is a $b^{'}$-cycle.*

Therefore Phi_{2n} is a cyclic homology cycle.

[**?**] shows that the cycles Phi_{2n} represent, up to a myltiplicative factors, the Pontrjagin classes of the bundle.

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