A quotient criterion for syzygies in equivariant cohomology

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Setup

Torus $T \cong (S^1)^r$ acts on the connected manifold X, $\dim X = n$.
Assume that $H^*(X)$ is finite-dimensional (with coefficients in $\mathbb{R}$).

equivariant i**-skeleton** $X_i = \{\, x \in X \mid \dim Tx \leq i \,\}$
Examples: $X_0 = X^T$, $X_r = X$

equivariant cohomology $H_T^*(X)$ f. g. module over
$A := H^*(BT) = \mathbb{R}[t_1, \ldots, t_r]$, $\deg t_i = 2$

Syzygies in equivariant cohomology

$H_T^*(X)$ torsion-free $/A$
$\Leftrightarrow \quad 0 \to H_T^*(X) \to H_T^*(X_0)$ exact

$H_T^*(X)$ reflexive $/A$
$\Leftrightarrow \quad 0 \to H_T^*(X) \to H_T^*(X_0) \xrightarrow{\delta} H_T^{*+1}(X_1, X_0)$ exact

$H_T^*(X)$ free $/A$
$\Leftrightarrow \quad H_T^*(X) \to H^*(X)$ surjective
$\Leftrightarrow \quad \dim H^*(X_0) = \dim H^*(X)$

Syzygies in equivariant cohomology

$H_T^*(X)$ torsion-free $/A$ $\Leftrightarrow$ $H_T^*(X)$ first syzygy $/A$
$\Leftrightarrow$ $0 \to H_T^*(X) \to H_T^*(X_0)$ exact

$H_T^*(X)$ reflexive $/A$ $\Leftrightarrow$ $H_T^*(X)$ second syzygy $/A$
$\Leftrightarrow$ $0 \to H_T^*(X) \to H_T^*(X_0) \xrightarrow{\delta} H_T^{*+1}(X_1, X_0)$ exact

$H_T^*(X)$ free $/A$ $\Leftrightarrow$ $H_T^*(X)$ r-th syzygy $/A$
$\Leftrightarrow$ $H_T^*(X) \to H^*(X)$ surjective
$\Leftrightarrow$ $\dim H^*(X_0) = \dim H^*(X)$

$H_T^*(X)$ is a ***k*-th syzygy** if $\exists$ $F_1, \ldots, F_k$ f. g. free $/A$ such that

$$0 \to H_T^*(X) \to F_1 \to \cdots \to F_{k-1} \to F_k \quad \text{exact}$$

Syzygies interpolate between torsion-freeness and freeness.

“Locally standard” actions

Assumption

T-action on X looks locally like action of $(S^1)^r$ on $\mathbb{C}^r \times \mathbb{R}^{\dim X - 2r}$.

For $\dim X = 2r$, this is called “locally standard”.

Then X/T is a manifold with corners.

rank of a face P of X/T = dimension of orbits over interior of P

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Proposition

If the above assumption is not satisfied (but $X_0 \neq \emptyset$), then one can correct this by ***blowing up*** *the non-free part of the action. This does not change the syzygy order of the equivariant cohomology.*

Previous results

Theorem (Masuda–Panov)

Assume that X *is compact orientable and* $\dim X = 2r$*. Then:*

$H_T^*(X)$ *free* $/A$ $\Leftrightarrow$ *all faces of* X/T *are* $\mathbb{R}$*-acyclic*

Previous results

Theorem (Masuda–Panov)

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Theorem (Goertsches–Rollenske)

Assume that X is compact. Then:

$H_T^*(X)$ *torsion-free* $/A \quad \Leftrightarrow \quad$ *the map*

$$\bigoplus_{Q \text{ facet of } P} H_*(Q) \to H_*(P)$$

is surjective for any face P of rank > 0

A generalization to all syzygies

For each face P of X/T, define a complex $\mathcal{B}^*(P)$ by

$$\mathcal{B}^i(P) = \bigoplus_{\substack{Q \text{ face of } P \\ \operatorname{rank} P = i}} H_*(Q)$$

with differential $d = \pm$ inclusion of facets.

Theorem

Let $0 \leq k \leq r$. Then: $H_T^*(X)$ is a k-th syzygy $\Leftrightarrow$ $H^i(\mathcal{B}^*(P)) = 0$ for any face P and any $i > \max(\operatorname{rank} P - k, 0)$.

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Corollary

Assume that X is compact orientable and $\dim X = 2r$. Then: $H_T^*(X)$ is free over A iff it is torsion-free.

References

C. Allday, M. Franz, V. Puppe
Equivariant cohomology, syzygies and orbit structure
Trans. Amer. Math. Soc. **366** (2014), 6567–6589

M. Franz
Big polygon spaces,
Int. Math. Res. Not. **2015** (2015), 13379–13405

M. Franz,
A quotient criterion for syzygies in equivariant cohomology,
Transformation Groups (2016), online

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  Int. Math. Res. Not. **2015** (2015), 13379–13405

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Thank you.