Cohomology and twisted K-theory of compact Lie groups

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Twisted K-theory of a compact space X is the K-theory of vector bundles over X twisted by a cocycle, and can be identified (non-canonically, but that won't matter for our purposes) with the K-theory of a stable continuous-trace algebra over X with given Dixmier-Douady class $h \in H^3(X)$. We will denote this $K^\bullet(X, h)$.

Physics Motivation

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The WZW Model (cont'd)

The WZW model is not a realistic string theory, but it is a favorite of theoretical physicists because it is "exactly solvable," and is closely related to the positive-energy representations of the central extension of the loop group $LG = C^\infty(S^1, G)$ with "central charge" related to the level.

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What's Known About $K^\bullet(G, h)$

What then is known about the twisted K-theory of compact simple Lie groups G? The simplest case is of course $G = \mathrm{SU}(2) \cong S^3$. In this case the twisted K-theory can be computed easily from the twisted Atiyah-Hirzebruch spectral sequence (AHSS)

$$H^p(X, K^q) \Rightarrow K^{p+q}(H, h)$$

which I developed back around 1981. Note that E_2 is the same as for ordinary K-theory and $K^q = 0$ for q odd, $\mathbb{Z}$ for q even; the twist $h \in H^3(X)$ only shows up in the differentials, the first of which is

$$d_3 = \mathrm{Sq}^3 + h \cup \_\colon\ H^p(X, K^q) \to H^{p+3}(X, K^{q-2}).$$

Thus for $X = \mathrm{SU}(2)$ there is only room for a single differential, and it's multiplication by h: $\mathbb{Z} = H^0(\mathrm{SU}(2)) \to H^3(\mathrm{SU}(2)) = \mathbb{Z}$.

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Surprise! Greg Moore, first by himself and then in collaboration with Maldacena and Seiberg, computed $K^\bullet(\mathrm{SU}(n+1), h)$, and the result turned out to be a surprise, even for $\mathrm{SU}(3)$:

$$K^\bullet(\mathrm{SU}(3), h) \cong \begin{cases} \mathbb{Z}/h, & h \text{ odd}, \\ \mathbb{Z}/(h/2), & h \text{ even}. \end{cases}$$

$K^\bullet(G, h)$

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Theorem (Hopkins-Braun-Douglas)

Let G be a simply connected compact simple Lie group and $h \neq 0$ a class in $H^3(G)$ (identified with an integer). Then $K^\bullet(G, h)$ is an exterior algebra over $\mathbb{Z}$ on $\operatorname{rank} G - 1$ (odd) generators, tensored with a finite cyclic group $\mathbb{Z}/h'$. Here $h' = h/\gcd(h, y)$, where y depends on G. For $G = \mathrm{SU}(n+1)$, $y = \operatorname{lcm}(1, 2, \cdots, n)$. For $G = G_2$, $y = 60$. For $G = F_4$ or E_6, $y = 27720$. For $G = E_8$, $y = 2329089562800$.

Statement of the Problem

We saw in the previous section that twists for K-theory take their values in H^3. That motivates wanting to understand this group. In particular, to relate twisted K-theory for a non-simply connected compact simple Lie group G with that of its universal cover $\widetilde{G}$, we need to understand the map $\pi^*\colon\ H^3(G) \to H^3(\widetilde{G})$ induced by the covering projection π. When dim $G = 3$, this is dual to the map π_* on top-degree homology, so it's just the degree of π. But this is only helpful in the single case of $G = \mathrm{SO}(3) = \mathrm{PSU}(2) = \mathrm{PSp}(1)$, where π is of degree 2. What happens in general?

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The answer is surprisingly complicated. But first an elementary observation: For G a compact simple Lie group, $H^3(G) \cong \mathbb{Z}$ except in the one case where $G = \mathrm{PSO}(4k)$, in which case $H^3(G) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. That's because $\pi_3(G) \cong \mathbb{Z}$, $\widetilde{G}$ is 2-connected, and this is the only case where $\pi_1(G)$ is not cyclic.

The map on H^3

Theorem (Mathai-Rosenberg)

Let G be a connected simple compact Lie group of rank n, with universal cover $\widetilde{G}$, with G not isomorphic to $\mathrm{PSO}(2n)$ with n even, and let d be the degree of the covering $\pi\colon \widetilde{G} \to G$.

1. *If G is of type A_n and if $n+1 = 2q$ with q odd, then π^* is multiplication by 2 if d is even and the identity if d is odd. If $n+1$ is either odd or divisible by 4, then π^* is the identity.*
2. *If G is of type C_n and π is not an isomorphism, i.e., $\widetilde{G} = \mathrm{Sp}(n)$ and $G = \mathrm{PSp}(n)$, then π^* is the identity if n is even and is multiplication by 2 if n is odd.*
3. *If $G = \mathrm{SO}(2n)$ and $\widetilde{G} = \text{Spin}(2n)$ with $n \geq 3$, or if $G = \mathrm{SO}(2n+1)$ and $\widetilde{G} = \text{Spin}(2n+1)$ with $n \geq 2$, or if $G = \mathrm{PSO}(2n)$ and $\widetilde{G} = \text{Spin}(2n)$ with $n \geq 3$ odd (so G is of type D_n or B_n, though we are excluding one other possibility for G in type D_n when n is even), then π^* is the identity.*
4. *If G is the adjoint group of E_6, then π^* is the identity, but if G is the adjoint group of E_7, then π^* is multiplication by $d = 2$.*

More on the Theorem

There is a similar result, which we'll omit here for simplicity, for the case $G = \mathrm{PSO}(4k)$. There is nothing to prove for the other simple Lie groups G_2, F_4, and E_8, because their compact forms are automatically simply connected.

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By a transfer argument π^* has to divide the degree d of the covering. We'll illustrate the method of proof for G a covering group of $\mathrm{PSU}(n+1)$.

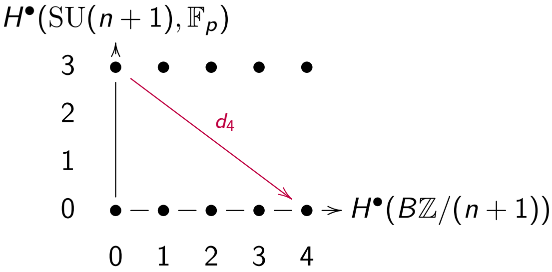
We have a factorization $\mathrm{SU}(n+1) \to G \to \mathrm{PSU}(n+1)$. So when $\pi^* = 1$ for the adjoint group $\mathrm{PSU}(n+1)$, then the same will be true for G as well. Next suppose that $n+1 = 2q$ with q odd and we know that $\pi^* = 2$ when $G = \mathrm{PSU}(n+1)$. Of the two coverings $\mathrm{SU}(n+1) \to G$ and $G \to \mathrm{PSU}(n+1)$, exactly one is of even degree. The only possibility is that π^* is 2 for the even covering and 1 for the odd covering. So the result is reduced to the case $G = \mathrm{PSU}(n+1)$.

Proof for $\mathrm{PSU}(n+1)$

The map $\pi^*\colon H^q(\mathrm{PSU}(n+1)) \to H^q(\mathrm{SU}(n+1))$ is an edge homomorphism for the Serre spectral sequence of the fibration

$$\mathrm{SU}(n+1) \to \mathrm{PSU}(n+1) \to B\mathbb{Z}/(n+1).$$

With $\mathbb{F}_p$ coefficients, p a prime dividing $n+1$, this looks like



Thus $\pi^*\colon H^3(\mathrm{PSU}(n+1)) \to H^3(\mathrm{SU}(n+1))$ is an isomorphism if and only if the transgression arrow d_4 vanishes.

Proof for $\mathrm{PSU}(n+1)$, cont'd

Now by Borel and Baum-Browder, if $n+1 = p^r m$, with p prime and $\gcd(m, p) = 1$, then

$$H^\bullet(\mathrm{SU}(n+1), \mathbb{F}_p) \cong \mathbb{F}_p \otimes \bigwedge(x_3, \cdots, x_{2n+1}),$$
$$H^\bullet(\mathrm{PSU}(n+1), \mathbb{F}_p) \cong \mathbb{F}_p[y]/(y^{p^r}) \otimes \bigwedge(x_1, x_3, \cdots, \widehat{x_{2p^r-1}}, \cdots, x_{2n+1}),$$

with y of degree 2, $\beta x_1 = y$ (β the Bockstein), and with the x_j's except for x_1 all reductions of integral classes. If $p = 2$ and $r = 1$, one has the additional relation $x_1^2 = y$. And the kernel of π^* is the ideal generated by x_1 and y.

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Langlands Duality

If G is a compact Lie group with maximal torus T, let's define its Langlands dual $G^\vee$ to be the unique compact Lie group obtained by interchanging $H^1(T)$ with $H_1(T)$ and switching the roots and coroots. For G simple of rank n, $G^\vee$ is locally isomorphic to G unless G is of type B_n or C_n with $n \geq 3$ (these two types are dual to each other). However, in all cases, G is simply connected $\Leftrightarrow$ $G^\vee$ is an adjoint group, and $(G^\vee)^\vee = G$.

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Theorem (Daenzer-Van Erp and Bunke-Nikolaus)

Assume G is a compact simple Lie group of rank n not of type B_n or C_n with $n \geq 3$. Then there are nonzero choices of $h \in H^3(G)$ and $h^\vee \in H^3(G^\vee)$ for which (G, h) and $(G^\vee, h^\vee)$ are T-dual as T^n-bundles over the flag variety $G/T \cong G^\vee/T^\vee$, and therefore $K^\bullet(G, h)$ and $K^\bullet(G^\vee, h^\vee)$ are isomorphic up to a degree shift.

Some Bad News

Unfortunately, the Bunke-Nikolaus recipe for choosing $h \in H^3(G)$ and $h^\vee \in H^3(G^\vee)$ in the last theorem was rather indirect, so it was hard to see immediately what $K^\bullet(G, h)$ works out to in this theorem. We had hoped at first that the Bunke-Nikolaus theorem would give an interesting example of calculation of twisted K-theory for nonsimply connected simple Lie groups.

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Theorem

In the situation just described, $K^\bullet(G, h) = K^\bullet(G^\vee, h^\vee) = 0$ *unless* rank $G = 1$*, where* $h = 2$*,* $h^\vee = 1$*, and*

$$K^0(\mathrm{SU}(2), h) = 0, \quad K^0(\mathrm{SO}(3), h^\vee) = \mathbb{Z}/2,$$
$$K^1(\mathrm{SU}(2), h) = \mathbb{Z}/2, \quad K^1(\mathrm{SO}(3), h^\vee) = 0.$$

Understanding the Twisted K-Theory

We still would like to try to understand the strange Braun-Douglas formula $h' = \frac{h}{\gcd(h,y)}$ for the order h' of the torsion in $K^\bullet(G, h)$. While this formula remains mysterious, we are at least able to give an elementary proof that h' and h coincide away from finitely many primes depending only on G (the primes dividing y).

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Here is an example:

Theorem

Suppose $h \in \mathbb{Z} \cong H^3(\mathrm{SU}(n+1))$ is relatively prime to $n!$. Then $K^\bullet(\mathrm{SU}(n+1), h)$ is $\mathbb{Z}/h$ tensored with an exterior algebra on $n-1$ odd generators.

Proof of the Theorem

We prove this by induction on n. To start the induction, we already know from the AHSS that $K^\bullet(\mathrm{SU}(2), h) \cong \mathbb{Z}/h$ (in odd degree). So assume $n \geq 2$ and the theorem is true for $n-1$, and look at the classical fibration

$$\mathrm{SU}(n) \to \mathrm{SU}(n+1) \to S^{2n+1}.$$

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Note that the inclusion $\mathrm{SU}(n) \to \mathrm{SU}(n+1)$ induces an isomorphism on H^3, so we get an induced map $K^\bullet(\mathrm{SU}(n+1), h) \to K^\bullet(\mathrm{SU}(n), h)$, as well as a Leray-type "Segal spectral sequence"

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If h is relatively prime to $n!$, then it is certainly relatively prime to $(n-1)!$, so the inductive hypothesis determines E_2.

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Theorem (Bott)

$\pi_{2n+1}(\mathrm{SU}(n+1)) \cong \mathbb{Z}$, $\pi_{2n}(\mathrm{SU}(n+1)) = 0$, *and* $\pi_{2n}(\mathrm{SU}(n)) \cong \mathbb{Z}/n!$.

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Thus the map $\pi_{2n+1}(\mathrm{SU}(n+1)) \to \pi_{2n+1}(S^{2n+1}) = \mathbb{Z}$ has image of index $n!$, and we have a splitting $S^{2n+1} \to \mathrm{SU}(n+1)$ away from primes dividing $n!$. Since $K^\bullet(\mathrm{SU}(n), h)$ is a direct sum of copies of $\mathbb{Z}/h$, the spectral sequence collapses and the theorem follows.

Additional Cases

Similar methodology applies in many other circumstances. Just to give an example with the exceptional groups, we have

Theorem

If h is not divisible by 2, 3, or 5, then $K^\bullet(G_2, h) \cong \mathbb{Z}/h$ in both even and odd degree.

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The Stiefel manifold $V_{7,2}$ is homotopy-equivalent to S^{11} after inverting 2. The homotopy sequence of the fibration now looks like

$$\pi_{11}(G_2) \to \pi_{11}(V_{7,2}) \to \pi_{10}(S^3) = \mathbb{Z}/15 \to \cdots .$$

So after inverting 2, 3, and 5, the Segal spectral sequence collapses. □

Level-Rank Duality

Twisted K-theory of compact groups is also related to what is often called level-rank duality or strange duality. This duality relates the WZW models on two different groups, for suitable values of the levels. For example, $\mathrm{SU}(n)$ at level k is dual to $\mathrm{SU}(k)$ at level n. In terms of twisted K-theory, this is often reflected in the order of the torsion in the twisted K-theories being the same. This is due to the fact that the correct twisting of the K-theory is given by $h = \ell + h^\vee(G)$, where $h^\vee(G)$ is the "dual Coxeter number," n for $\mathrm{SU}(n)$. So for example, $\mathrm{SU}(3)$ at level 4 and $\mathrm{SU}(4)$ at level 3 both give twisted K-groups with torsion of order $4 + 3 = 7$.

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