Superconnections and a Finslerian Gauss-Bonnet-Chern formula

Huitao Feng (Joint with Ming Li)

Chern Institute of Mathematics, Tianjin, China

2016-08-10, Kraków
Glances@Manifolds II

Outline

In this talk, we will establish a Finslerian Gauss-Bonnet-Chern formula for an even dimensional closed and oriented Finsler manifold (M, F).

A brief review of Finsler geometry

- M: C^∞ manifold of dim. $2n$; $\pi : TM \to M$ the tangent bundle of M.
- Given a local coordinate chart $(U; (x^i))$ of M, there is a corresponding local coordinate chart $(\pi^{-1}(U), (x^i, y^i))$ of TM.
- (M, F) is called a Finsler manifold if F is a C^∞ function $F : TM_0 = TM \setminus 0 \to \mathbb{R}^+$ with the properties:
  (i) $F(x, \lambda y) = \lambda F(x, y), \forall \lambda \in \mathbb{R}^+$;
  (ii) $(g_{ij}(x, y)) = \left(\frac{1}{2}\left[F^2\right]_{y^i y^j}\right) > 0$ on TM_0.
- So $g_{ij}(x, y)$ are homogenous functions of degree zero on y, and so all geometric quantities coming from g_{ij} actually live on the unit sphere bundle SM of (M, F).

- Using $(g_{ij}(x,y))$ one can introduce a natural Euclidean structure on the pull-back bundle $\pi^* TM \to TM_0$.
- The Euclidean vector bundle $(\pi^* TM, g)$ is essential in the study of Finsler geormetry.
- There are two connections often used in Finsler geo.: one is the Chern conn. ∇^{Ch}, and the other is the Cartan conn. ∇^{Car}.
- On the other hand, the Finsler metric F determines canonically a horizontal subbundle $H(TM_0)$ of $T(TM_0)$ and the identification:

$$H(TM_0) \cong \pi^* TM.$$

- let $V(TM_0)$ denote the natural vertical subbundle of $T(TM_0)$.
- There is a canonical isomorphism $J: H(TM_0) \cong V(TM_0)$, by which and the iso. $H(TM_0) \cong \pi^* TM$, the space TM_0 admits a well-defined Riemannian metric $g^{T(TM_0)}$.
- As a natural foliated manifold $(TM_0, V(TM_0))$, it is well-known that there is a Bott connection $\widetilde{\nabla}^{H(TM_0)}$ on $H(TM_0)$, which is flat along the leaves in TM_0, that is

$$\left(\widetilde{\nabla}^{H(TM_0)}\right)^2 (U, V) = 0$$

for any $U, V \in V(TM_0)$.

Proposition 1 (F-Li, CAG, 2013)

Under the identification $H(TM_0) \cong \pi^* TM$.

- *the Bott connection $\widetilde{\nabla}^{H(TM_0)}$ on $H(TM_0)$ is exactly the Chern connection ∇^{Ch} on $\pi^* TM$;*
- *the symmetrization $\widehat{\nabla}^{H(TM_0)}$ of the Bott connection turns out to be the Cartan connection ∇^{Car}.*

Remark 1

Let $R^{\mathrm{Ch}} = (\nabla^{\mathrm{Ch}})^2$ be the curvature of Chern connection. By the flatness along leaves, R^{Ch} splits only into two parts

$$R^{\mathrm{Ch}} = R + P,$$

where R is an $\mathrm{End}(\pi^* TM)$-valued horizontal two-form on TM_0, and P is an $\mathrm{End}(\pi^* TM)$-valued horizontal-vertical two-form on TM_0.

A biref review of the GBC Formula in Riem. geom.

- In 1943, S.S. Chern gave a simple and intrinsic proof of the following Gauss-Bonnet-Chern formula for closed Riemannian manifold (M, g^{TM}) of even dimension:

Proposition 2 (S.S. Chern)

$$\chi(M) = \left(\frac{-1}{2\pi}\right)^n \int_M \mathrm{Pf}(R^{TM}),$$

where R^{TM} is the curvature of the Levi-Civita connection ∇^{TM} associated to the Riemannian metric g^{TM} on M, and the Pfaffian $\mathrm{Pf}(R^{TM})$ is a well-defined $2n$-form on M constructed by the curvature R^{TM} of ∇^{TM}.

- Chern's remarkable transgression formula on SM

$$\left(\frac{-1}{2\pi}\right)^n \pi^* \mathrm{Pf}(R^{TM}) = -d^{SM}\Pi.$$

- Let $X \in \Gamma(TM)$ with the isolated zero point set $Z(X)$. Then on $M_\epsilon = M \setminus Z_\epsilon(X)$,

$$\left(\frac{-1}{2\pi}\right)^n \mathrm{Pf}(R^{TM}) = -[X]^* d^{SM}\Pi = -d^M [X]^*\Pi.$$

- By using the Poincaré-Hopf theorem, he obtained

$$\left(\frac{-1}{2\pi}\right)^n \int_M \mathrm{Pf}(R^{TM}) = \lim_{\epsilon \to 0} \int_{\partial Z_\epsilon(X)} [X]^*\Pi = \chi(M).$$

Generalizations in the Finsler setting before

- Many people have made important contributions to the generalizations of the Chern's formula in the Finsler setting:

  Lichnerowicz, Bao-Chern, Z. Shen, Lackey, $\cdots$.

Lichnerowicz's Work(1949)

- Lichnerowicz constructed an analogue differential $2n$-form $\mathrm{Pf}(R^{\mathrm{Car}})$ on SM and proved a transgression formula

$$\mathrm{Pf}(\Omega^{\mathrm{Car}}) = -d^{SM}\Pi^{\mathrm{Car}},$$

  for some $2n-1$-form Π^{Car} on SM, where R^{Car} denotes the curvature of the Cartan connection on π^*TM.
- Following Chern's strategy, Lichnerowicz also made the computations

$$\int_M [X]^*\mathrm{Pf}(\Omega^{\mathrm{Car}}) := \lim_{\epsilon\to 0}\int_{M_\epsilon}[X]^*\mathrm{Pf}(\Omega^{\mathrm{Car}}) = \lim_{\epsilon\to 0}\int_{\partial Z_\epsilon(X)}[X]^*\Pi^{\mathrm{Car}}.$$

To get the desired Euler number $\chi(M)$ from the above computations, Lichnerowicz needed the following assumptions:

- The space (M, F) should be a Cartan-Berwald space.
- All Finsler unit spheres $S_xM = \{Y \in T_xM | F(Y) = 1\}$ should have the same volume as a Euclidean unit sphere

Note that the later assumption holds automatically for all Cartan-Berwald spaces of dimension larger than 2.

Bao and Chern's Work(Ann. of Math. 1996)

Around fifty years later, Bao and Chern dropped the Cartan-Berwald condition of Lichnerowicz by using the Chern connection ∇^{Ch}.

- They proved a transgression formula

$$\mathrm{Pf}(\Omega^{\mathrm{Ch}}) + \mathcal{F} = -d^{SM}\Pi^{\mathrm{Ch}},$$

  where $\mathrm{Pf}(R^{\mathrm{Ch}})$ is some $2n$-form on SM constructed from the curvature R^{Ch} of the Chern connection ∇^{Ch}, $\mathcal{F}$ is a correction term and Π^{Ch} is the associated transgression form.
- Then they established the following GBC-formula for all $2n$-dimensional oriented and closed Finsler manifolds with the constant volume of Finsler unit spheres:

$$\int_M [X]^*[\mathrm{Pf}(\Omega^{\mathrm{Ch}}) + \mathcal{F}] = \chi(M)\frac{\mathrm{Vol}(\mathrm{Finsler}S^{n-1})}{\mathrm{Vol}(S^{n-1})}$$

Shen and Lackey's works

- The constant volume condition in Bao-Chern's formula is a much strong assumption for a Finsler manifold.
- To avoid this defect, following Bao-Chern's approach, Z. Shen and Lackey independently modified the GBC-integrand terms by using the unit sphere volume function $V(x) = \mathrm{Vol}(S_x M)$ and obtained some new types of GBC-formulae respectively via the Cartan and the Chern connections for all oriented and closed Finsler manifolds.
- Roughly, their formulae are of the form:

$$\chi(M) = \mathrm{Vol}(S^{n-1}) \int_M \frac{[X]^*[\mathrm{Pf}(\Omega^{\mathrm{Ch}}) + \widetilde{\mathcal{F}}]}{\mathrm{Finsler}(S^{n-1})}$$

Remark

- A striking difference from the Chern's formula for Riemannian manifolds, all generalizations of the GBC formula in the Finsler setting before had to make use of an extra vector field X on M to obtain the related GBC-integrands.

- In a recent paper of Prof. Y. Shen, he asked explicitly whether there is a GBC-type formula for general Finsler manifolds without using any vector fields.

In the following, we try to answer Prof. Shen's question by using a Mathai-Quillen-type formula obtained by Weiping Zhang and myself recently.

A Finslerian GBC formula

Main theorem (F-Li)

Let (M, F) be a closed and oriented Finsler manifold of dimension $2n$. Then for any connection ∇ on $\pi : TM \to M$, one has

$$\chi(M) = \frac{1}{(2\pi)^{2n}(2n)!} \left\{ \sum_{k=1}^{n} C_{2n}^{2k} \int_{SM} \mathrm{tr}_s \left[c(\mathbf{e}) c(\nabla^{\mathrm{Ch}} \mathbf{e})^{2k-1} (R^{\natural})^k (P^{\natural})^{2n-2k} \right] + \int_{SM} \mathrm{tr}_s \left[\theta^{\natural} \left(P^{\natural} \right)^{2n-1} \right] \right\},$$

where $\mathbf{e} = \hat{Y}|_{SM}$ and $R^{\natural}$, $P^{\natural}$, $\theta^{\natural}$ are the natural lifting of R, P, θ respectively on $\Lambda^*(\pi^* T^* M)$, and θ is defined by $\theta =: \nabla^{\mathrm{Ch}} - \pi^* \nabla$. In particular, the formula is independent of the choice of the connection ∇ on TM.

A Mathai-Quillen-type Formula

- In a recent work of Weiping Zhang and myself, we obtained a Mathai-Quillen-type formula on the Euler number $\chi(M)$.
- More precisely, we applied the Mathai-Quillen's geometric construction of the Thom class to the pull-back exterior algebra bundle $\pi^*\Lambda^*(T^*M) \to TM$ and obtained an integrable top-form on TM from any connection ∇ on TM, and proved that the integration of this top form over TM is the Euler characteristic $\chi(M)$.
- Here the key point is that the connections used needn't metric-preserving!

In the following, we will recall briefly this Mathai-Quillen-type formula.

- Let ∇^{TM} be any connection on TM. Then it induces a connection $\nabla^{\Lambda^*(T^*M)}$ on the exterior algebra bundle $\Lambda^*(T^*M)$, which preserves the even/odd $\mathbf{Z}_2$-grading in $\Lambda^*(T^*M)$.
- Let $\hat{Y}$ denote the tautological section of the pull-back bundle π^*TM:

$$\hat{Y}(x, Y) := Y \in (\pi^*TM)|_{(x,Y)},$$

  where $(x, Y) \in TM$ with $x \in M$ and $Y \in T_xM$.
- For any given Euclidean metric g^{TM} on TM, let $\hat{Y}^*$ denote the dual of $\hat{Y}$ with respect to the pull-back metric π^*g^{TM} on π^*TM.

- Then the Clifford action $c(\hat{Y}) = \hat{Y}^* \wedge -i_{\hat{Y}}$ acts on $\pi^*\Lambda^*(T^*M)$ and exchanges the even/odd grading in $\pi^*\Lambda^*(T^*M)$. Moreover,

$$c(\hat{Y})^2 = -|\hat{Y}|^2_{\pi^* g^{TM}} = -|Y|^2_{g^{TM}}.$$

- For any $T > 0$, define the superconnection

$$A_T = \pi^*\nabla^{\Lambda^*(T^*M)} + Tc(\hat{Y})$$

on the bundle $\pi^*\Lambda^*(T^*M)$.

Then we have the following Mathai-Quillen-type formula:

Theorem 1 (F-Zhang, 2016)

$$\chi(M) = \left(\frac{1}{2\pi}\right)^{2n} \int_{TM} \mathrm{tr}_s[\exp(A_T^2)].$$

A slight generalization of Theorem 1

One can choose any connection ∇ with the curvature $R = \nabla^2$ bounded along fibres of TM and any Euclidean metric g on the pull-back bundle $\pi^* TM$ to define a superconnection on $\pi^*\Lambda^*(T^*M) \equiv \Lambda^*(\pi^* T^*M)$.

- Let $\nabla^{\Lambda^*(\pi^* T^* M)}$ be the lifting of the connection ∇ on $\Lambda^*(\pi^* T^* M)$;
- Let $\hat{Y}^*_g$ denote the dual of $\hat{Y}$ with respect to the metric g and set $c_g(\hat{Y}) = \hat{Y}^*_g \wedge - i_{\hat{Y}}$;
- For any $T > 0$,

$$\tilde{A}_T = \nabla^{\Lambda^*(\pi^* T^* M)} + T c_g(\hat{Y}) \tag{1}$$

is also a superconnection on $\pi^*\Lambda^*(T^*M)$.

Corollary 1

Let M be a closed and oriented manifold of dimension $2n$. Then for any connection ∇ and any Euclidean metric g on $\pi^* TM$, if the curvature $R = \nabla^2$ is bounded along fibres of TM, then the following formula holds for any $T > 0$:

$$\chi(M) = \left(\frac{1}{2\pi}\right)^{2n} \int_{TM} \mathrm{tr}_s[\exp(\tilde{A}_T^2)],$$

where $\tilde{A}_T$ is defined by (1).

Sketch Proof of Corollary 1

- Set for any $T > 0$ and $t \in [0,1]$,

$$\omega_T = \tilde{A}_T - A_T = \nabla^{\Lambda^*(\pi^* T^* M)} - \pi^* \nabla^{\Lambda^*(T^* M)} + T(\hat{Y}_g^* - \hat{Y}^*)\wedge,$$

$$A_{T,t} = t\tilde{A}_T + (1-t)A_T = \pi^* \nabla^{\Lambda^*(T^* M)} + t\omega_T + Tc(\hat{Y}).$$

- Since the curvature $R = \nabla^2$ is bounded along fibres of TM, one verifies that $\mathrm{tr}_s \left[\omega_T \exp(A_{T,t}^2)\right]$ is exponentially decay along fibres of $\pi : TM \to M$, and so $\int_{TM/M} \int_0^1 \mathrm{tr}_s \left[\omega_T \exp(A_{T,t}^2)\right] dt$ is a well-defined differential form on M.

- Therefore,

$$
\begin{aligned}
&\int_{TM} \mathrm{tr}_s \left[\exp(\tilde{A}_T^2) \right] - \int_{TM} \mathrm{tr}_s \left[\exp(A_T^2) \right] \\
= &\int_{TM} d^{TM} \int_0^1 \mathrm{tr}_s \left[\omega_T \exp(A_{T,t}^2) \right] dt \\
= &\int_M \int_{TM/M} d^{TM} \int_0^1 \mathrm{tr}_s \left[\omega_T \exp(A_{T,t}^2) \right] dt \\
= &\int_M d^M \int_{TM/M} \int_0^1 \mathrm{tr}_s \left[\omega_T \exp(A_{T,t}^2) \right] dt \\
= &0.
\end{aligned}
$$

Sketch Proof of Main Theorem

- In the following we will use Corollary 1 to establish a Finslerian Gauss-Bonnet-Chern formula, in which no extra vector field is involved.
- In particular, an explicit GBC-type integrand on M will be given in the induced homogeneous coordinate charts on SM.

Main theorem (F-Li)

Let (M, F) be a closed and oriented Finsler manifold of dimension $2n$. Then for any connection ∇ on $\pi : TM \to M$, one has

$$\chi(M) = \frac{1}{(2\pi)^{2n}(2n)!}\left\{\sum_{k=1}^{n} C_{2n}^{2k}\int_{SM}\mathrm{tr}_s\left[c(\mathbf{e})c(\nabla^{\mathrm{Ch}}\mathbf{e})^{2k-1}(R^{\natural})^{k}(P^{\natural})^{2n-2k}\right] + \int_{SM}\mathrm{tr}_s\left[\theta^{\natural}\left(P^{\natural}\right)^{2n-1}\right]\right\},$$

where $\mathbf{e} = \hat{Y}|_{SM}$ and $R^{\natural}$, $P^{\natural}$, $\theta^{\natural}$ are the natural lifting of R, P, θ respectively on $\Lambda^*(\pi^*T^*M)$, and θ is defined by $\theta =: \nabla^{\mathrm{Ch}} - \pi^*\nabla$. In particular, the formula is independent of the choice of the connection ∇ on TM.

Sketch Proof of Main Theorem

Step 1. Extended Chern connections

- Let $D_r M = \{(x, Y) \in TM | F(x, Y) < r\}$ be the r-disc bundle for $\forall r > 0$.
- Let $\rho \in C^\infty(TM)$ such that $0 \le \rho \le 1$, and

$$\rho = \begin{cases} 0, & \text{on } TM \setminus D_{\frac{1}{2}} M; \\ 1, & \text{on } D_{\frac{1}{4}} M. \end{cases}$$

- The extended Chern connection $\widetilde{\nabla}^{\mathrm{Ch}}$ is defined on $\pi^* TM \to TM$,

$$\widetilde{\nabla}^{\mathrm{Ch}}_\rho = (1 - \rho)\nabla^{\mathrm{Ch}} + \rho\pi^*\nabla.$$

- We similarly get a smooth metric g on $\pi^* TM \to TM$ from the fundamental tensor g_F on TM_0.

Step 2.

- For any $T > 0$, we define the following superconnection

$$\tilde{A}_{\rho,T} = \widetilde{\nabla}_{\rho}^{\mathrm{Ch},\natural} + Tc_{\tilde{g}_F}(\hat{Y}).$$

- Note that $\lim_{T\to+\infty} \exp(-T^2\|Y\|^2) = 0$ for $F(Y) \geq 1$, we have

$$\begin{aligned}\chi(M) &= \left(\frac{1}{2\pi}\right)^{2n} \int_{TM} \mathrm{tr}_s \left[\exp(\tilde{A}_{\rho,T}^2)\right] \\ &= \lim_{T\to+\infty} \left(\frac{1}{2\pi}\right)^{2n} \int_{D_1 M} \mathrm{tr}_s \left[\exp(\tilde{A}_{\rho,T}^2)\right].\end{aligned}$$

Step 3.

Note that $\mathrm{tr}_s\left[\exp(\pi^*\nabla^{\Lambda^*(T^*M)})^2\right] = 0$, we have

$$\int_{D_1M}\mathrm{tr}_s\left[\exp(\tilde{A}^2_{\rho,T})\right] = \int_{D_1M}\mathrm{tr}_s\left[\exp(\tilde{A}^2_{\rho,T})\right] - \int_{D_1M}\mathrm{tr}_s\left[\exp((\widetilde{\nabla}^{\mathrm{Ch},\natural}_{\rho})^2)\right]$$
$$+\int_{D_1M}\mathrm{tr}_s\left[\exp((\widetilde{\nabla}^{\mathrm{Ch},\natural}_{\rho})^2)\right] - \int_{D_1M}\mathrm{tr}_s\left[\exp(\pi^*\nabla^{\Lambda^*(T^*M)})^2\right].$$

Step 4.

The first transgression:

$$\int_{D_1 M} \mathrm{tr}_s \left[\exp(\tilde{A}^2_{\rho,T}) \right] - \int_{D_1 M} \mathrm{tr}_s \left[\exp((\widetilde{\nabla}^{\mathrm{Ch},\natural}_{\rho})^2) \right]$$

$$= \int_{SM} \int_0^1 i^* \mathrm{tr}_s \left[Tc(Y) \exp(\nabla^{\mathrm{Ch}} + Ttc(Y))^2 \right] dt$$

$$= \sum_{k=1}^{n} \frac{C_{2n}^{2k}}{(2n)!} \int_{SM} \mathrm{tr}_s \left[c(\mathbf{e}) c(\nabla^{\mathrm{Ch}} \mathbf{e})^{2k-1} (R^{\natural})^k (P^{\natural})^{2n-2k} \right]$$

Step 5.
Set

$$\theta_\rho = \widetilde{\nabla}^{\mathrm{Ch}}_\rho - \pi^*\nabla.$$

The second transgression:

$$\int_{D_1 M} \mathrm{tr}_s \left[\exp((\widetilde{\nabla}^{\mathrm{Ch},\natural}_\rho)^2)\right] - \int_{D_1 M} \mathrm{tr}_s \left[\exp(\pi^*\nabla^{\Lambda^*(T^*M)})^2\right]$$

$$= \int_{SM}\int_0^1 i^*\mathrm{tr}_s \left[\theta^\natural_\rho \exp(\nabla_t)^2\right] dt$$

$$= \int_{SM}\int_0^1 \mathrm{tr}_s \left[\theta^\natural \exp\left(R^{\mathrm{Ch},\natural} - (1-t)[\nabla^{\mathrm{Ch},\natural}, \theta^\natural] + (1-t)^2\theta^\natural \wedge \theta^\natural\right)\right] dt.$$

Note that

- $$[\nabla^{\mathrm{Ch},\natural}, \theta^{\natural}] = \theta^{\natural} \wedge \theta^{\natural} - \left(\pi^* \nabla^{\Lambda^*(T^*M)}\right)^2 + R^{\mathrm{Ch},\natural}.$$

- The term $\theta^{\natural}$ is an $\mathrm{End}(\Lambda^*(\pi^* T^* M))$-valued horizontal one form, and so
$$tR^{\natural} + (1-t)(\pi^* \nabla^{\Lambda^*(T^*M)})^2 - t(1-t)\theta^{\natural} \wedge \theta^{\natural}$$
is an $\mathrm{End}(\Lambda^*(\pi^* T^* M))$-valued horizontal two form.

$$
\begin{aligned}
&\int_{D_1 M}\left(\mathrm{tr}_s\left[\exp((\widetilde{\nabla}_{\rho}^{\mathrm{Ch},\natural})^2)\right] - \mathrm{tr}_s\left[\exp((\pi^*\nabla^{\Lambda^*(T^*M)})^2)\right]\right)\\
=&\int_{SM}\int_0^1 \mathrm{tr}_s\left[\theta^\natural \exp\left(tP^\natural + tR^\natural + (1-t)(\pi^*\nabla^{\Lambda^*(T^*M)})^2 - t(1-t)\theta^\natural\wedge\theta^\natural\right)\right] dt\\
=&\int_{SM}\int_0^1 \mathrm{tr}_s\left[\theta^\natural \exp\left(tP^\natural\right)\right] dt\\
=&\int_{SM}\frac{1}{(2n-1)!}\int_0^1 t^{2n-1}dt\ \mathrm{tr}_s\left[\theta^\natural (P^\natural)^{2n-1}\right]\\
=&\frac{1}{(2n)!}\int_{SM}\mathrm{tr}_s\left[\theta^\natural\left(P^\natural\right)^{2n-1}\right].
\end{aligned}
$$

In the homogeneous coordinate charts (x^i, y^i) on SM, we have:

Corollary 2 (F-Li)

Let (M, F) be a closed and oriented Finsler manifold of dimension $2n$. Let $R^{\mathrm{Ch}} = R + P$ be the curvature of the Chern connection ∇^{Ch} on the pull-back bundle π^*TM over SM. Then in the induced homogeneous coordinate charts (x^i, y^i) on SM, one has

$$\chi(M) = \frac{1}{(2\pi)^{2n}(2n)!}\left\{\sum_{k=1}^{n}(-1)^k C_{2n}^{2k} C_{2k-2}^{k-1}\int_{SM}\delta^{i_1\ldots i_{2n}}_{j_1\ldots j_{2n}} R^{j_1}_{i_1}\cdots R^{j_k}_{i_k} P^{j_{k+1}}_{i_{k+1}}\cdots P^{j_{2n-k}}_{i_{2n-k}}\cdot \Upsilon^{j_{2n-k+1}}_{i_{2n-k+1}}\cdots\Upsilon^{j_{2n-1}}_{i_{2n-1}}\Xi^{j_{2n}}_{i_{2n}} + \int_M\int_{SM/M}\delta^{i_1\ldots i_{2n}}_{j_1\ldots j_{2n}} P^{j_1}_{i_1}\cdots P^{j_{2n-1}}_{i_{2n-1}}\varpi^{j_{2n}}_{i_{2n}}\right\},$$

where ϖ^j_i, R^j_i, P^j_i, Υ^j_i and Ξ^j_i are defined respectively in the following:

- (ϖ^j_i) is the connection matrix of the Chern connection ∇^{Ch} with respect to natural frames on $\pi^* TM$;
- (R^j_i), (P^j_i) are the curvature matrices of the R-part, P-part of the curvature $(\nabla^{\mathrm{Ch}})^2$, respectively;
- $$\left(\nabla^{\mathrm{Ch}}\mathbf{e}\right)^i := d\left(\frac{y^i}{F}\right) + \frac{y^j}{F}\varpi^i_j, \quad (\nabla^{\mathrm{Ch},*}\omega)_i := dF_{y_i} - F_{y^j}\varpi^j_i,$$
- $$\Upsilon^j_i := (\nabla^{\mathrm{Ch},*}\omega)_i \left(\nabla^{\mathrm{Ch}}\mathbf{e}\right)^j;$$
$$\Xi^j_i := \left(F_{y^i}(\nabla^{\mathrm{Ch}}\mathbf{e})^j - \frac{y^j}{F}(\nabla^{\mathrm{Ch},*}\omega)_i\right);$$
- $\mathbf{e} = Y/F(Y)$ and $\omega = \mathbf{e}^*$ is the Hilbert form.

Sketch Proof of Corollary 2

- Let (x^i, y^i) be any homogeneous coordinate charts on SM.
- Set

$$\nabla^{\mathrm{Ch}} \frac{\partial}{\partial \hat{x}^j} = \varpi^i_j \otimes \frac{\partial}{\partial \hat{x}^i}, \quad \nabla \frac{\partial}{\partial x^j} = \vartheta^i_j \otimes \frac{\partial}{\partial x^i}$$

- Set

$$R \frac{\partial}{\partial \hat{x}^j} = R^i_j \otimes \frac{\partial}{\partial \hat{x}^i}, \quad P \frac{\partial}{\partial \hat{x}^j} = P^i_j \otimes \frac{\partial}{\partial \hat{x}^i}.$$

- We have

$$\varpi^\natural = -\varpi_i^j d\hat{x}^i \wedge i_{\frac{\partial}{\partial \hat{x}^j}}, \quad \vartheta^\natural = -\vartheta_i^j d\hat{x}^i \wedge i_{\frac{\partial}{\partial \hat{x}^j}}$$

$$R^\natural = -R_i^j d\hat{x}^i \wedge i_{\frac{\partial}{\partial \hat{x}^j}}, \quad P^\natural = -P_i^j d\hat{x}^i \wedge i_{\frac{\partial}{\partial \hat{x}^j}}.$$

In particular,

$$P_j^i = P_j{}^i{}_{kl} dx^k \wedge \frac{\delta y^l}{F} = -dx^k \wedge \left(\frac{\partial \Gamma_{jk}^i}{\partial y^l} \delta y^l \right).$$

- Denote that

$$c(\mathbf{e}) = \omega \wedge - i_{\mathbf{e}} = F_{y^j} d\hat{x}^j \wedge - \frac{y^i}{F} i_{\frac{\partial}{\partial \hat{x}^i}},$$

where $\omega = \mathbf{e}^* = F_{y^i} d\hat{x}^i$ is the Hilbert form on SM;

$$\left(\nabla^{\mathrm{Ch}}\mathbf{e}\right)^i := d\left(\frac{y^i}{F}\right) + \frac{y^j}{F}\varpi^i_j, \quad (\nabla^{\mathrm{Ch},*}\omega)_i := dF_{y_i} - F_{y^j}\varpi^j_i,$$

$$\Upsilon^j_i := (\nabla^{\mathrm{Ch},*}\omega)_i \left(\nabla^{\mathrm{Ch}}\mathbf{e}\right)^j, \quad \Xi^j_i := \left(F_{y^i}(\nabla^{\mathrm{Ch}}\mathbf{e})^j - \frac{y^j}{F}(\nabla^{\mathrm{Ch},*}\omega)_i\right).$$

We have

- $$\begin{aligned}
&\mathrm{tr}_s\left[c(\mathbf{e})c(\nabla^{\mathrm{Ch}}\mathbf{e})^{2k-1}(R^{\natural})^k(P^{\natural})^{2n-2k}\right]\\
=&\mathrm{tr}_s\left[\left(-R_i^j d\hat{x}^i\wedge i_{\frac{\partial}{\partial\hat{x}^j}}\right)^k\left(-P_i^j d\hat{x}^i\wedge i_{\frac{\partial}{\partial\hat{x}^j}}\right)^{2n-2k}\right.\\
&\left((\nabla^{\mathrm{Ch},*}\omega)_j d\hat{x}^j\wedge-(\nabla^{\mathrm{Ch}}\mathbf{e})^i i_{\frac{\partial}{\partial\hat{x}^i}}\right)^{2k-2}\\
&\left.\left((\nabla^{\mathrm{Ch},*}\omega)_p d\hat{x}^p\wedge-(\nabla^{\mathrm{Ch}}\mathbf{e})^q i_{\frac{\partial}{\partial\hat{x}^q}}\right)\left(F_{y^l}d\hat{x}^l\wedge-\frac{y^r}{F}i_{\frac{\partial}{\partial\hat{x}^r}}\right)\right]\\
=&(-1)^k C_{2k-2}^{k-1}\delta_{j_1\dots j_{2n}}^{i_1\dots i_{2n}}R_{i_1}^{j_1}\cdots R_{i_k}^{j_k}P_{i_{k+1}}^{j_{k+1}}\cdots P_{i_{2n-k}}^{j_{2n-k}}\Upsilon_{i_{2n-k+1}}^{j_{2n-k+1}}\cdots\Upsilon_{i_{2n-1}}^{j_{2n-1}}\Xi_{i_{2n}}^{j_{2n}}.
\end{aligned}$$

- $$\begin{aligned}&\mathrm{tr}_s\left[\theta^{\natural}\left(P^{\natural}\right)^{2n-1}\right]\\=&\mathrm{tr}_s\left[-\left(\varpi_i^j-\pi^*\vartheta_i^j\right)d\hat{x}^i\wedge i_{\frac{\partial}{\partial\hat{x}^j}}\left(-P_k^l d\hat{x}^k\wedge i_{\frac{\partial}{\partial\hat{x}^l}}\right)^{2n-1}\right]\\=&\delta_{j_1\dots j_{2n}}^{i_1\dots i_{2n}}P_{i_1}^{j_1}\cdots P_{i_{2n-1}}^{j_{2n-1}}\left(\varpi_{i_{2n}}^{j_{2n}}-\pi^*\vartheta_{i_{2n}}^{j_{2n}}\right).\end{aligned}$$

- Note that P_i^j are vertical exact, we get

$$\int_{SM/M}P_{i_1}^{j_1}\cdots P_{i_{2n-1}}^{j_{2n-1}}(\pi^*\vartheta_{i_{2n}}^{j_{2n}})=\vartheta_{i_{2n}}^{j_{2n}}\int_{SM/M}P_{i_1}^{j_1}\cdots P_{i_{2n-1}}^{j_{2n-1}}=0.$$

- Therefore, we get

$$
\begin{aligned}
&\frac{1}{(2n)!}\int_{SM}\mathrm{tr}_s\left[\theta^{\natural}\left(P^{\natural}\right)^{2n-1}\right] \\
=&\frac{1}{(2n)!}\int_{M}\int_{SM/M}\mathrm{tr}_s\left[\theta^{\natural}\left(P^{\natural}\right)^{2n-1}\right] \\
=&\frac{1}{(2n)!}\int_{M}\int_{SM/M}\delta^{i_1\cdots i_{2n}}_{j_1\cdots j_{2n}}P^{j_1}_{i_1}\cdots P^{j_{2n-1}}_{i_{2n-1}}\varpi^{j_{2n}}_{i_{2n}}
\end{aligned}
$$

Berwald spaces

- The P-part of the Chern curvature vanishes for Berwald spaces.
- With respect to any orthonormal frame $\{e_1, \cdots, e_{2n}\}$ of $\pi^* TM$ with $e_{2n} = \mathbf{e}$, set

$$\nabla^{\mathrm{Ch}} e_i = \omega_i^j e_j, \quad R^{\mathrm{Ch}} e_i = R e_i = R_i^j e_j,$$

and so

$$R^{\mathrm{Ch},\natural} = -R_i^j e^{*,i} \wedge i_{e_j} = -\frac{1}{4} R_i^j (\hat{c}(e_i) + c(e_i))(\hat{c}(e_j) - c(e_j)).$$

Corollary 3

Let (M, F) *be a closed and oriented Berwald space. Then one has,*

$$\chi(M) = \left(\frac{-1}{2\pi}\right)^n \frac{1}{\mathrm{Vol}(S^{n-1})} \int_M \int_{SM/M} \mathrm{Pf}(\widehat{R}^{\mathrm{Ch}}) \omega_1^{2n} \cdots \omega_{2n-1}^{2n},$$

where $\widehat{R}_i^j = \frac{(R_i^j - R_j^i)}{2}$ *and*

$$\mathrm{Pf}(\widehat{R}^{\mathrm{Ch}}) = \frac{1}{2^n n!} \sum_{i_1,\ldots,i_{2n}=1}^{2n} \epsilon_{i_1,\ldots,i_{2n}} \widehat{R}_{i_1}^{i_2} \wedge \cdots \wedge \widehat{R}_{i_{2n-1}}^{i_{2n}}.$$

Berwald surfaces

Combining with Bao-Chern's result for closed and oriented Berwald surfaces:

$$\int_M -R_1{}^2{}_{12}\omega^1 \wedge \omega^2 = \chi(M)\mathrm{Vol}(\mathrm{Finsler}\,S^1),$$

we get

Corollary 4

Let (M, F) *be a closed and oriented Berwald surface, then*

$$\left[\mathrm{Vol}(\mathrm{Finsler}\,S^1) - 2\pi\right]\chi(M) = 0, \tag{2}$$

that is, $\mathrm{Vol}(\mathrm{Finsler}\,S^1) = 2\pi$, *or* $\chi(M) \neq 0$.

In fact, by the Szabó's rigidity theorem that any Berwald surfaces must be locally Minkowskian or Riemannian, and a closed locally Minkowskian surface has zero Euler number, one also gets (2) easily.

Thank You!