Categorical approach to equivariant Morse theory

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Morse theory

Let M be C^∞, compact, Riemannian manifold (without boundary), and

$$f : M \longrightarrow \mathbb{R}$$

a C^∞ map. A critical point $p \in M$ is *Morse* if the bilinear form

$$Hess_p(f) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right)$$

is non-degenerate.

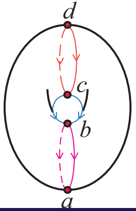
The gradient flow lines $\gamma : \mathbb{R} \longrightarrow M$ satisfying

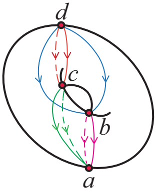
$$\frac{d\gamma}{dt} + \nabla_\gamma(f) = 0 .$$

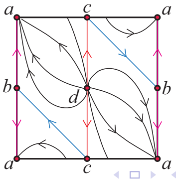
For *a* critical point, the *stable* and *unstable* manifold are

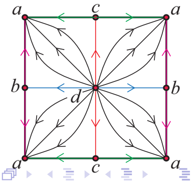
$$W^s(a) = \{x \in M : \lim_{t\to+\infty} \gamma_x(t) = a\}$$

$$W^u(a) = \{x \in M : \lim_{t\to-\infty} \gamma_x(t) = a\}$$









Classifying spaces

Definition

For $\mathcal{C}$ a category, the classifying space $B\mathcal{C}$ is the realization of the nerve $N\mathcal{C}$. Where for a simplicial set (space) X the realization is defined as the quotient $\amalg_{n\geq 0}\Delta_n \times X_n / \sim$ with $(s, X(f)a) \sim (\Delta_f(s), a)$ and we get:

- Category $\mathcal{C} \longmapsto$ Topological space $B\mathcal{C}$
- Functor $F : \mathcal{C} \to \mathcal{D} \longmapsto$ Continuous function $BF : B\mathcal{C} \to B\mathcal{D}$
- Natural transformation $\alpha : F \Rightarrow G \longmapsto$ Homotopy $H_\alpha : B\mathcal{C} \times I \to B\mathcal{D}$

Definition

Let $F : \mathcal{C} \to \mathcal{D}$ functor,y in $\mathcal{D}$. The category $y \setminus F$ has objects (x, v), $v : y \to F(x)$, morphisms from (x, v) to (x', v') is $u : x \to x'$, $v' = F(u)v$.

Theorem A (Quillen)

If the category $y \setminus F$ is contractible for every object y of $\mathcal{D}$, then the functor F is a homotopy equivalence.

Morse theory and classifying spaces

For $f : M \to \mathbb{R}$ a Morse function we define the "*flow category*" $\mathcal{C}_f$ as follows:

- the objects, are just the union of all the critical points

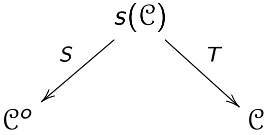
$$\mathrm{Obj}\,\mathcal{C}_f = \coprod_{p \in \mathrm{Crit}_f} p$$

- for two critical points a and b we define the space of objects $\mathrm{Hom}_{\mathcal{C}_f}(a, b)$ as the compactification of the moduli space $\mathcal{M}(a, b) = (W^u(a) \cap W^s(b))/\mathbb{R}$. We denote this space by $\overline{\mathcal{M}}(a, b)$.

Theorem (Cohen-Jones-Segal)

- *For $f : M \longrightarrow \mathbb{R}$ a Morse function, the classifying space of $\mathcal{C}_f$ is of the homotopy type of M.*
- *For $f : M \longrightarrow \mathbb{R}$ a Morse-Smale function, the classifying space of $\mathcal{C}_f$ is homeomorphic with M.*

- For a category $\mathcal{C}$ there are pair of functors



where $s(\mathcal{C})$ has objects $a \xrightarrow{\gamma} b$ and morphism from $a_1 \xrightarrow{\gamma_1} b_1$ to $a_2 \xrightarrow{\gamma_2} b_2$ pairs $a_2 \xrightarrow{\alpha} a_1$ and $b_1 \xrightarrow{\beta} b_2$ with $\gamma_2 = \beta\gamma_1\alpha$. This functors are prefibred and

$$S^{-1}(x) = x \setminus C,\ T^{-1}(y) = (C/y)^o$$

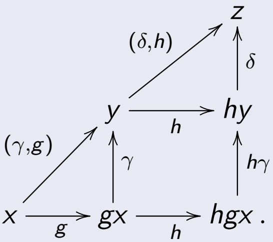
- There is a projection functor for the flow category $\overline{s}(\mathcal{C}_f) \to s(\mathcal{C}_f)$ with $\overline{s}(\mathcal{C}_f)$ the category with pais (γ, x) as objects with x in γ and morphism as in $s(\mathcal{C}_f)$ but with the same x. The induced map in n-chains has contractible fiber, so it is a homotopy equivalence.
- Let $\underline{M}$ the category with objects the elements of M and morphism only identities, so $B\underline{M} = M$. There are functors $\underline{M} \leftrightarrows \overline{s}(\mathcal{C}_f)$, defined by $x \longmapsto (\gamma_x, x)$ and projection. This categories are homotopy equivalent.

Semi-direct product

Definition

Suppose G acts on $\mathcal{C}$, <u>the semi-direct product</u> $\mathcal{C} \rtimes G$ is a category with:

- the objects of $\mathcal{C}$;
- the morphisms are pairs $(\gamma, g) : x \longrightarrow y$ with $g \in G$ and $\gamma : gx \longrightarrow y$ a morphism in $\mathcal{C}$; and
- the composition of $(\gamma, g) : x \longrightarrow y$ with $(\delta, h) : y \longrightarrow z$ is $(\delta h\gamma, hg)$. This is described as follows.



Theorem

The classifying space of the semi-direct product $\mathcal{C} \rtimes G$ has the weak homotopy type of the Borel construction $B\mathcal{C} \times_G EG$.

Theorem A (Quillen-Moerdijk)

Let $F : \mathcal{D} \longrightarrow \mathcal{C}$ be a G-invariant continuous functor between topological categories. If for $n \geq 0$, the quotient map

$$B\left(\mathrm{Nerve}_n(\mathcal{C})\diagdown F\right)/G \longrightarrow \mathrm{Nerve}_n(\mathcal{C})$$

is a weak homotopy equivalence, then

$$\widehat{BF} : B\mathcal{D}/G \longrightarrow B\mathcal{C}$$

is a weak homotopy equivalence.

- For $F : \mathcal{D} \longrightarrow \mathcal{C}$ functor and $\varphi : X \longrightarrow \mathcal{C}_0$ continuous map, with $\mathcal{C}_0$ the objects. The objects of $X \diagdown F$ are triples (x, u, y) with $x \in X$, $y \in \mathcal{D}_0$ and $u : \varphi(x) \longrightarrow F(y)$; the morphisms $\gamma : (x, u, y) \longrightarrow (x', u', y')$ for $x = x'$ are arrows $\gamma : y \longrightarrow y'$ with $F(\gamma) \circ u = u'$.
- Let $\overline{G}$ be the category with objects G and only one morphism between any pair of objects. Consider the functor $T : \mathcal{C} \times \overline{G} \to \mathcal{C} \rtimes G$ defined in objects $(x, g) \longmapsto g^{-1}x$ and in morphisms $(x, g) \overset{(\gamma, h^{-1}g)}{\longrightarrow} (y, h)$ the image by T is $(h^{-1}\gamma, h^{-1}g)$. Denote the category $\mathcal{T} := \mathrm{Nerve}_n(\mathcal{C} \rtimes G)/T$
- $$B\mathcal{T} = \coprod_{\mathrm{Nerve}_n(\mathcal{C} \rtimes G)} B\mathcal{T}_{\overline{x}} \simeq \coprod_{\mathrm{Nerve}_n(\mathcal{C} \rtimes G)} \coprod_{k \in G} B\mathcal{T}_k \cong \coprod_{\mathrm{Nerve}_n(\mathcal{C} \rtimes G)} \coprod_{g \in G} EG$$

  where we have the action relates $\mathcal{T}_k \overset{g}{\to} \mathcal{T}_{gk}$ and the inclusion $\mathcal{T}_k \hookrightarrow \overline{G}$ is a homotopy equivalence. Thus $B\mathcal{T}/G$ is of the (weak) homotopy type of $\mathrm{Nerve}_n(\mathcal{C} \rtimes G)$ and hence $B(\mathcal{C} \times \overline{G})/G \simeq B(\mathcal{C} \rtimes G)$.

Equivariant Morse category

Let M be a compact manifold with an action of a Lie group G, that is

$$M \times G \longrightarrow M\,.$$

Furthermore, if N_1, N_2 are two Morse submanifolds, then we have an action

$$G \times W(N_1, N_2) \times \mathbb{R} \longrightarrow W(N_1, N_2)$$

given by $(g, x, t) \longrightarrow g\gamma_x(t)$ where suppose $g\gamma_x = \gamma_{gx}$ as sets. Thus we have an action of G over the flow category $\mathcal{C}_f$ and we get the following result.

Theorem

- *For a G-invariant Morse function we get*

$$B(\mathcal{C}_f \rtimes G) \simeq B\mathcal{C}_f \times_G EG\,.$$

- *For G a finite group we get*

$$B(\mathcal{C}_f \rtimes G) \simeq B\mathcal{C}_f \times_G EG \simeq B(B\mathcal{C}_f \rtimes G)\,.$$

Corollary

For G a group acting free over a manifold M and $f : M \longrightarrow \mathbb{R}$ a G-invariant function, we get the (weak) homotopy equivalence

$$B(\mathcal{C}_f \rtimes G) \simeq M/G\,.$$

Thanks!!