Some constructions on K-contact manifolds

Bogusław Hajduk

Uniwersity of Warmia and Mazury in Olsztyn

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A 1-form η on an oriented manifold M^{2n+1} is **contact** if $\eta \wedge (d\eta)^n > 0$ et every point.
Locally,

$$\eta = dz_0 + x_1\, dy_1 + x_2\, dy_2 + ... + x_n\, dy_n$$

in some local coordinates $z_0, x_1, ..., x_n, y_1, ..., y_n$.

The Reeb field of η is the non-zero vector field ξ defined by $\eta(\xi) = 1, d\eta(\xi, *) = 0$.

The subbundle $ker\,\eta$ admits (many) complex structures compatible with η, i.e. an endomorphisms J such that $J^2 = -Id;\ d\eta(JX, JY) = d\eta(X, Y);\ d\eta(JX, X) > 0$ for $X \neq 0$.
Any such J defines an endomorphism Φ of TM such that $\Phi|ker\,\eta = J, \Phi(\xi) = 0$.
Also, $g(X, Y) = d\eta(\Phi(X), Y) + \eta(X)\eta(Y)$ satisfies $g(X, Y) = d\eta(JX, Y)$ if $X, Y \in ker\,\eta$, $g(\xi, \xi) = 1, g(\xi, Y) = 0$ if $Y \in ker\,\eta$, thus g is a Riemannian metric.

Let (M, η) be a contact manifold. If J is compatible with η, then we say that (M, η, J) is ***K-contact*** if the Reeb field ξ of η is a Killing vector field with respect to the Riemannian metric $g(X, Y) = d\eta(\Phi(X), Y) + \eta(X)\eta(Y)$.

For a contact manifold (M, η), its ***symplectization*** (sometimes called metric cone) is the symplectic manifold

$$\mathcal{C}(M) = (M \times \mathbb{R}_{>0}, d(t^2\eta)), \quad t \in \mathbb{R}.$$

Given a K-contact manifold (M, η, J), the almost complex structure I on $\mathcal{C}(M)$ is defined by:

1. $I(X) = J(X)$ on $\operatorname{Ker} \eta$;
2. $I(\xi) = t\frac{\partial}{\partial t}, I(t\frac{\partial}{\partial t}) = -\xi$.

A K-contact manifold is called ***Sasakian***, if the almost complex structure I is integrable, hence defines a Kähler structure on $\mathcal{C}(M)$.
The Reeb field of a K-contact manifold defines (1-dimensional) transversally symplectic foliation; for a Sasakian manifold the transverse structure is Kähler.

[Rukimbira] Any K-contact structure on a closed manifold can be approximated by a K-contact structure whose Reeb flow generates a circle action (called **quasi - regular** K-contact structure). The same for Sasakian structures.

WHAT ARE TOPOLOGICAL PROPERTIES OF CLOSED K-CONTACT AND SASAKIAN MANIFOLDS?
(here by K-contact (Sasakian) manifold we mean a manifold which admits a K-contact (Sasakian) structure)

Corollary. If a closed manifold admits a K-contact structure, then it also admits a fixed point free circle action.

Let (M, η, J) be a quasi-regular K-contact (resp. Sasakian) structure.
Then the quotient $X = M/\xi$ is a cyclic orbifold; symplectic for K-contact structure and Kähler for Sasakian structure.

(since the Reeb foliation is transversally symplectic (transversally Kähler)).

Example: Boothby - Wang manifolds. Let (B, ω) be a symplectic manifold such that $[\omega]$ is an integer class. Let M be the total space of the circle bundle over B with $\omega \in c_1$. Then M is K-contact and fibers are orbits of the Reeb field.

$$\{Sasakian\ \ manifolds\} \subset \{K - contact\ \ manifolds\} \subset \{contact\ \ manifolds\}$$

$\{Sasakian\ \ manifolds\} \subsetneq \{K - contact\ \ manifolds\} \subsetneq \{contact\ \ manifolds\}$

For non-simply connected non - Sasakian manifolds, see [**Boyer, Galicki: Sasakian Geometry**]

Simply connected K-contact non - Sasakian examples, dim > 7 [BH, Tralle, 2013]
Simply connected K-contact non - Sasakian examples, dim = 7 [Muñoz, Tralle, 2014]
Simply connected K-contact 5-manifold M with $H_1(M, \mathbb{Z}) = 0$ having no Sasakian structure without isolated singular orbits of the Reeb flow [Muñoz, Rojo, Tralle, 2015]

Simply connected closed 5-manifolds are classified by $H_2(*, \mathbb{Z})$ and the second Stiefel -Whitney class w_2. [Smale 1962, Barden 1965]

[Geiges]: A simply connected closed M^5 admits a contact structure iff it admits an almost contact structure (complex structure on the linear complement of a non-zero vector field) iff w_2 is the mod 2 reduction of an integer class.

[Kollar] Characterization of simply connected closed 5-manifolds which admit fixed point free actions.
(for any prime $p > 2$ the minimal number of generators of the $p-$torsion part of $H_2(M, \mathbb{Z})$ is less or equal to $b_2(M) + 1$ and another condition on 2-torsion and w_2)

Let (M, η, J) be a quasi-regular K-contact, M_p be the sum of orbits of ξ with isotropy $\mathbb{Z}_p$ for a prime p.

Proposition. Any connected component of M_p is a Seifert 2-manifold such that the quotient is a J-holomorphic submanifold $\Sigma \subset M/\xi$. In particular, cohomology classes of these components $C = [\Sigma]$ satisfy the adjunction formula $2g(\Sigma) = C \cdot C - c_1(M/\xi, J) \cdot C + 2$. Moreover, if M is simply connected, p-torsion of M is $\oplus \mathbb{Z}_p^{2g(\Sigma)}$, the sum over all components of M_p.

Reduction of torsion.

Consider a tubular neighborhood N of a connected component of M_p. Its boundary is a torus (orbifold) bundle over Σ and orbits of ξ are curves of type $(1, p)$ in fibres.

Proposition. There exists an orbifold bundle $E \to \Sigma$ with fibre solid torus $S^1 \times D^2$ over Σ with a fibrewise linear diffeomorphism $\partial E \to \partial N$ such that the flow of ξ extends to a circle action on E, free outside singular fibres.

Corollary. Any simply connected closed K-contact manifold with nontrivial torsion in H_2 is obtained from another such manifold having less torsion by cutting-and-pasting construction described above.

Thanks!