Elliptic problems associated with diffeomorphisms of manifolds with boundary

Anton Savin and Boris Sternin

Peoples' Friendship University of Russia
and
Leibniz Universität Hannover

10.08.2016

Operators associated with group actions

- M is a closed smooth manifold;
- G is a discrete group of diffeomorphisms $g: M \to M$;
- $\Rightarrow$ induced action of G on functions on M by shift operators $T_g u(x) = u(g^{-1}(x))$.

G-pseudodifferential operators

$$D = \sum_g D_g T_g : H^s(M) \longrightarrow H^{s-m}(M) \qquad (1)$$

1. D_g are (pseudo)differential operators on M of orders $\leq m$;
2. $H^s(M)$ and $H^{s-d}(M)$ are Sobolev spaces.

Problem: construct elliptic theory for operators (1).

Differential operators on NC torus (Connes)

On the real line with coordinate x consider operators

$$D=\sum_{\alpha+\beta\le m} a_{\alpha\beta}x^\alpha\left(-i\frac{d}{dx}\right)^\beta,$$

where $a_{\alpha\beta}$ are Laurent polynomials in operators

$$(V_1u)(x)=u(x+\theta),\qquad (V_2u)(x)=e^{2\pi ix}u(x).$$

V_1 and V_2 do not commute: $V_1V_2=e^{2\pi i\theta}V_2V_1$.

G-operators on closed manifolds

Main results:

- Finiteness theorem (Antonevich and Lebedev);
- Index theorems (Nazaikinskii-Savin-Sternin, Savin-Sternin, Perrot);
- related studies in noncommutative geometry, e.g., conformal spectral triple, noncommutative torus, noncommutative orbifolds,... (Connes and Moscovici, Ponge and Wang, ...)

Generalizations

G-operators are also studied

in relative elliptic theory, i.e., for pairs (M, X), where X is a submanifold in M of arbitrary codimension.

for Lie groups G (relations with integro-differential equations and transversally-elliptic operators).

on manifolds with singularities.

...

In this work we consider G-operators on manifolds with boundary.

1. Statement of problem

Statement of the problem

1. Geometric setup.

- W is a closed smooth manifold;
- $M \subset W$ is a smooth submanifold with boundary;
- $g : W \to W$ is a diffeomorphism.

Definition

Diffeomorphism g is a contraction if $g(M) \subset M \setminus \partial M$.

Example

$M = \mathbb{B}^n \subset \mathbb{R}^n$, $g(x) = qx$, $0 < q < 1$.

2. Shift operators for contractions.

If $g : M \to M$ is a contraction, then we have shift operators

$$u(x) \stackrel{T^{-1}}{\longmapsto} u(gx) \qquad u(x) \stackrel{T}{\longmapsto} \begin{cases} u(g^{-1}x) & \text{if } g^{-1}x \in M; \\ 0 & \text{otherwise.} \end{cases}$$

- Clearly, operator T^{-1} acts on $C^\infty(M)$ and expands supports of functions.
- On the contrary, operator T contracts supports of functions and does not act on $C^\infty(M)$.
- One shows that:
  - operator $T^{-1} : H^s(M) \to H^s(M)$ is bounded for all s;
  - operator $T : H^s(M) \to H^s(M)$ is bounded for $|s| < 1/2$.

Statement of problem

3. We consider boundary value problems of the form

$$\begin{pmatrix} D \\ i^*B \end{pmatrix} : H^s(M) \longrightarrow \begin{matrix} H^{s-m}(M) \\ \oplus \\ H^{s-b-1/2}(\partial M) \end{matrix}, \qquad |s-m|<1/2,$$

where $i^* : H^s(M) \to H^{s-1/2}(\partial M)$ is the restriction operator and

$$D = \sum_k T^k D_k, \qquad B = \sum_{k\le 0} T^k B_k.$$

Here D_k and B_k are differential operators on M.

Problem: obtain Fredholm property for nonlocal boundary value problems of the form (D, B).

2. Fredholm property

Analysis of the problem

- to define the symbol use localization (freezing of coefficients);
- since the problem is nonlocal, we have to freeze coefficients on the trajectories of points in M;
- for a contraction $g : M \to M$ we have 3 types of trajectories:
  1. trajectories $\{g^n x\}$, which are contained in M for all $n \in \mathbb{Z}$;
  2. trajectories $\{g^n x\}$, which are contained in M only for $n \geq 0$;
  3. trajectories $\{g^n x\}$ of points $x \in \partial M$ on the boundary.

We compute the symbol of the boundary value problem for these 3 types of trajectories.

Symbol for orbits $\{g^n x\}, n \in \mathbb{Z}$

We want to define the symbol $\sigma(D)(x,\xi)$ at an arbitrary point $(x,\xi) \in T^*M \setminus 0$ of the cotangent bundle.

The symbol should involve the entire orbit $\{\partial g^n(x,\xi)\}_{n\in\mathbb{Z}}$, where

$$\partial g : T^*M \longrightarrow T^*M$$

stands for the codifferential of g.

Definition (trajectory symbol)

The symbol of $D = \sum_k T^k D_k : H^s(M) \to H^{s-m}(M)$ is the operator

$$\sigma(D)(x,\xi) : l^2(\mathbb{Z}, \mu_{x,\xi,s}) \longrightarrow l^2(\mathbb{Z}, \mu_{x,\xi,s-m}),$$

$$\sigma(D)(x,\xi)u(n) = \sum_k \mathcal{T}^k \sigma(D_k)\Big(\partial g^n(x,\xi)\Big)\Big(u\Big)(n).$$

here

- $l^2(\mathbb{Z}, \mu_{x,\xi,s})$ is the space of functions $u = \{u(n)\}$ on the trajectory, μ is a certain measure on $\mathbb{Z}$;
- symbol of T is shift on the group $\mathcal{T}(n) = u(n-1)$;
- symbol of D_k is operator of multiplication: $\sigma(D_k)(\partial g^n(x,\xi))$.

$$\mu_{x,\xi,s}(k) = \frac{(\partial g^k)^*(\mu\sigma(\Delta)^s)(x,\xi)}{\mu(x)} \equiv \left|\det \frac{\partial g^k}{\partial x}\right| \cdot \left|{}^t\!\left(\frac{\partial g^k}{\partial x}\right)^{-1}(\xi)\right|^{2s}.$$

Here μ is a volume form on M and Δ is a Laplacian.

Symbol for orbits $\{g^n x\}, n \geq 0$

We want to define the symbol $\sigma(D)(x,\xi)$ at an arbitrary point $(x,\xi) \in T^*M \setminus 0$ such that $g^n(x) \notin M$ for all $n < 0$.

$\Rightarrow$ the symbol is defined as on the previous slide, but here the trajectory is isomorphic to $\mathbb{Z}_+ = \{n \geq 0\}$

$\Rightarrow$ we obtain the symbol of the form:

Definition (trajectory symbol)

$$\sigma(D)(x,\xi) : l^2(\mathbb{Z}_+, \mu_{x,\xi,s}) \longrightarrow l^2(\mathbb{Z}_+, \mu_{x,\xi,s-m}),$$

Symbol for orbits of boundary points

We want to define the symbol $\sigma(D)(x,\xi)$ at an arbitrary point $(x,\xi) \in T^*M \setminus 0$ such that $x \in \partial M$.

Freezing the coefficients at the points of the orbit and making Fourier transform along the variables tangent to the boundary we obtain an operator-valued function

Boundary trajectory symbol

$$\sigma(D,B)(x,\eta): \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ H^s(\mathbb{R}\times\mathbb{Z}_{>0},\mu_{x,\eta,s}) \end{matrix} \longrightarrow \begin{matrix} H^{s-m}(\mathbb{R}_+) \\ \oplus \\ H^{s-m}(\mathbb{R}\times\mathbb{Z}_{>0},\mu_{x,\eta,s-m}) \\ \oplus \\ \mathbb{C}, \end{matrix} \tag{2}$$

A direct computation shows that this symbol acts as

$$\sigma(D,B)(x,\eta) =$$

$$\begin{pmatrix} \sigma(D_0)\left(x,\eta,-i\frac{d}{dt}\right) & (\partial g)^*\sigma(D_{-1})\left(x,\eta,-i\frac{d}{dt}\right) & \cdots \\ \sigma(D_1)\left(x,\eta,-i\frac{d}{dt}\right) & (\partial g)^*\sigma(D_0)\left(x,\eta,-i\frac{d}{dt}\right) & \cdots \\ \vdots & \vdots & \ddots \\ \sigma(B_0)\left(x,\eta,-i\frac{d}{dt}\right)\bullet\Big|_{t=0} & (\partial g)^*\sigma(B_{-1})\left(x,\eta,-i\frac{d}{dt}\right)\bullet\Big|_{t=0} & \cdots \end{pmatrix}$$

Here $D=\sum_k T^k D_k$, $B=\sum_{k\le 0} T^k B_k$

$$(u_0, u_1, u_2, \ldots) \overset{\sigma(D,B)(x,\eta)}{\longmapsto} (v_0, v_1, v_2, \ldots, w)$$

$$v_n = \sum_{k \le n} \Big[(\partial g^{n-k})^* \sigma(D_k) \Big] \left(x, \eta, -i\frac{d}{dt} \right) u_{n-k}, \quad n \ge 0,$$

$$w = \left(\sum_{k \le 0} \Big[(\partial g^{-k})^* \sigma(B_k) \Big] \left(x, \eta, -i\frac{d}{dt} \right) u_{-k} \right) \Bigg|_{t=0}$$

Ellipticity. Finiteness theorem

Definition

Boundary value problem (D, B) is elliptic if

- its interior trajectory symbol $\sigma(D)(x,\xi)$ is invertible for all $(x,\xi) \in T^*M \setminus 0$;
- (analogue of Shapiro–Lopatinskii condition) and its boundary trajectory symbol $\sigma(D,B)(x',\eta)$ is invertible for all $(x',\eta) \in T^*\partial M \setminus 0$.

Theorem (Finiteness)

Elliptic boundary value problem (D, B) is Fredholm in Sobolev spaces

$$(D,B) : H^s(M) \to H^{s-m}(M) \oplus H^{s-b-1/2}(\partial M).$$

Idea of the proof

1. To deal with infinite orbits use theory of C^*-algebras.
2. To apply C^*-algebras use order reduction (Rempel and Schulze).
3. To deal with singularities use theory of transmission problems due to Rempel and Schulze.

Note here that for $f \in C^\infty(M)$ the operator TfT^{-1} is an operator of multiplication by a function with a jump at the submanifold $g(\partial M) \subset M$.

$\Rightarrow$ almost-inverse operators in our theory belong to an operator algebra generated by shift operators and pseudodifferential operators with transmission conditions on the infinite sequence of submanifolds $g^n(\partial M) \subset M$, $n \geq 1$.

3. Example

Example

Consider the boundary value problem

$$(D,B)=\Big((aT+b+cT^{-1})\Delta, i^*\Big): H^s(M)\longrightarrow H^{s-2}(M)\oplus H^{s-1/2}(X), \tag{3}$$

where M is the unit ball in $\mathbb{R}^k$ with center at the origin, $X=\partial M$ is the unit sphere, shift operators T and T^{-1} are defined by the contraction $x\mapsto qx$, $0<q<1$. Here a,b,c are some constants. Finally, we suppose that $|s-2|<1/2$.

This boundary value problem is a composition of Dirichlet problem for the Laplacian and the operator

$$R=aT+b+cT^{-1}: H^{s-2}(M)\longrightarrow H^{s-2}(M).$$

Let us find conditions, under which the latter operator is elliptic. To this end we compute its trajectory symbols.

1. Symbol at the fixed point $x = 0$. A direct computation shows that $\mu_{x,\xi,s}$ is equal to

$$\mu_{x,\xi,s}(n) = \frac{\partial g^{n*}[\mu\sigma(\Delta^s)](x,\xi)}{\mu(x)} = q^{n(\dim M - 2s)}.$$

Hence, we have

$$\sigma(R) = a\mathcal{T} + b + c\mathcal{T}^{-1} : l^2(\mathbb{Z}, q^{n(\dim M - 2(s-2))}) \longrightarrow l^2(\mathbb{Z}, q^{n(\dim M - 2(s-2))})$$

where $\mathcal{T}u(n) = u(n-1)$ is the shift of sequences. Using Fourier series, we see that this operator is isomorphic to the operator of multiplication by the function

$$az + b + cz^{-1} \tag{4}$$

on the circle of radius $q^{(\dim M - 2(s-2))/2}$. Ellipticity condition: the function in (4) is nonzero on the circle.

2. The trajectory symbol in the domain $q < |x| < 1$ is equal to

$$\sigma(R) = \pi(a\mathcal{T} + b + c\mathcal{T}^{-1}) : \\ l^2(\mathbb{Z}_+, q^{n(\dim M-2(s-2))}) \longrightarrow l^2(\mathbb{Z}_+, q^{n(\dim M-2(s-2))})$$

where π is the projection on the subspace of sequences equal to zero on $\mathbb{Z}_-$. This operator is isomorphic to the Toeplitz operator

$$\Pi_+(az + b + cz^{-1}) : \mathrm{Im}\,\Pi_+ \longrightarrow \mathrm{Im}\,\Pi_+,$$

on the Hardy space $\mathrm{Im}\,\Pi_+ \subset L^2(\mathbb{S}^1, |z| = q^{(\dim M-2(s-2))/2})$ in the disc of radius $q^{(\dim M-2(s-2))/2}$.
Ellipticity condition: this Toeplitz operator is invertible.

A direct computation shows that this occurs precisely, when $az^2 + bz + c$ has precisely one zero inside the disc $|z| \leq q^{(\dim M-2(s-2))/2}$.

3. A direct computation shows that the boundary trajectory symbol is invertible, provided that the ellipticity conditions in 1. and 2. are satisfied.
We obtain the Fredholm property for our boundary value problem.

Theorem

Boundary value problem (3) is Fredholm if precisely one solution of the equation

$$az^2 + bz + c = 0$$

lies inside the circle $|z| = q^{\dim M - 2(s-2)}$.

$$\operatorname{ind}(D, B) = 0.$$

4. Homotopy classification problem

Stable homotopies

Definition

Two elliptic operators D_0 and D_1 are stably homotopic if there exists a homotopy

$$D_0 \oplus D_0^{triv} \simeq D_1 \oplus D_1^{triv}$$

such that

- homotopy consists of elliptic operators;
- D_0^{triv}, D_1^{triv} are (invertible) operators — bundle isomorphisms.

We obtain equivalence relation and Abelian group:

$$\mathrm{Ell}(M) = \{\text{elliptic operators on } M\} / \sim$$

Old results

Theorem (Classification on closed manifolds)

If M is compact and closed, then

$$\mathrm{Ell}(M) \simeq K_c^0(T^*M) \simeq K_0(M).$$

Theorem (Classification of boundary value problems)

If M is compact and with boundary ∂M, then

$$\mathrm{Ell}(M) \simeq K_c^0(T^*(M \setminus \partial M)) \simeq K_0(M).$$

Corollary (Atiyah and Bott)

Given elliptic operator D on M, there exists (stably) elliptic boundary value problem (D, B) if and only if $\sigma(D)$ is homotopic to identity at the boundary.

$$
\begin{array}{ccccc}
\mathrm{Ell}(M) & \longrightarrow & \mathrm{Ell}(M \setminus \partial M) & & \\
\downarrow \simeq & & \downarrow \simeq & & \\
K_c^0(T^*(M \setminus \partial M)) & \longrightarrow & K_c^0(T^*M) & \longrightarrow & K_0^1(T^*\partial M)
\end{array}
$$

Recent result

G — amenable group;
G acts on M
$\Rightarrow$ group $\mathrm{Ell}(M, G)$ of st. homotopy classes of elliptic G-problems

Theorem (Savin and Sternin, 2016)

$$\mathrm{Ell}(M, G) \simeq K_0(C_0(T^*(M \setminus \partial M)) \rtimes G),$$

where $C_0(T^*(M \setminus \partial M)) \rtimes G$ stands for the crossed product of the algebra of continuous function on $T^*(M \setminus \partial M)$ vanishing at infinity and group G.

Corollary

1. If G is finite, then $\mathrm{Ell}(M, G) \simeq K_0^G(M)$;
2. If $G = \mathbb{Z}^n$, then $\mathrm{Ell}(M, \mathbb{Z}^n) \simeq K_0(M \times \mathbb{R}^n/\mathbb{Z}^n)$.

Thank you very much for your attention!