Property (T) and Higher Index Theory (by Piotr Nowak)

Plain sailing: property (T) for $Sl(n, \mathbb{K})$

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Exercise 1. Generating set for $Sl(n, \mathbb{K})$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$, or any different non-discrete field). Let δ_{ij} denote the matrix which differs from the zero matrix by 1 in the i, j-th entry. For $x \in \mathbb{K}$ denote by $E_{ij}(x)$ the matrix

$$E_{ij}(x) = I_n + x\delta_{ij}.$$

- By elementary arguments show that $Sl(2, \mathbb{K})$ is generated by matrices $E_{12}(x)$ and $E_{21}(x)$. This stays true for coefficients in any Euclidean domain.
- Extend those arguments to $Sl(n, \mathbb{K})$ and matrices $E_{ij}(x)$ for $1 \leqslant i, j \leqslant n$ and $i \neq j$.

Definition. Denote by

$$\begin{aligned} N^+ &= \{E_{12}(x) \text{ for } x \in \mathbb{K}\} < Sl(2, \mathbb{K}) \\ N^- &= \{E_{21}(x) \text{ for } x \in \mathbb{K}\} < Sl(2, \mathbb{K}) \\ A &= \left\{ \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} \text{ for } x \in \mathbb{K}^\times \right\} \subset Sl(2, \mathbb{K}) \end{aligned}$$

Exercise 2 (Mautner's Lemma). Let $(\pi, \mathcal{H})$ be a *unitary* representation of a topological group G. Suppose that there exists a sequence $(y_i)_i \in G$ such that $\lim_i y_i x y_i^{-1} = e$. Prove that if $\xi \in \mathcal{H}$ is (y_i)-invariant then it is x-invariant.

Exercise 3. We say that a subgroup $H < G$ is **characteristic** if (for all unitary representations $(\pi, \mathcal{H})$) whenever $\mathcal{H}$ contains an H-invariant vector, it also has G-invariant vector.

- Let $(\lambda_i)_i$ be a net in $\mathbb{K}$ such that $\lambda_i \neq 0$ and $\lim_i \lambda_i = 0$. Prove that

$$\lim_i a_i N^+ a_i^{-1} = \lim_i a_i^{-1} N^- a_i = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

  where $a_i = \begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{bmatrix} \in Sl(2, \mathbb{K})$.

- Prove that $\xi \in \mathcal{H}$ is A-invariant then it is N^+-invariant (N^--invariant).

- Prove that if $\xi \in \mathcal{H}$ is N^+-invariant it is A-invariant. (Tip: consider function $\varphi(g) = \langle \pi(g)\xi, \xi \rangle$)
- Prove that N^+ is characteristic for $Sl(2, \mathbb{K})$.
  - What about standard left-matrix-multiplication action of $Sl(2, \mathbb{K})$ on $\mathbb{K}^2$?
- Prove that any standard embedding of $Sl(2, \mathbb{K}) \hookrightarrow Sl(n, \mathbb{K})$ is characteristic (start with $n = 3$, use the standard generators and Mautner's Lemma).

Definition (Property (T) for pairs (G, H)). Let $H < G$ be a subgroup. We say that **pair** (G, H) **has property (T)** if and only if (for all unitary representations $(\pi, \mathcal{H})$ of G) if $\mathcal{H}$ contains almost G-invariant vectors, then $\mathcal{H}$ contains a non-zero H-invariant vector.

Exercise 4. Let $n \geq 3$ and consider the subgroups of $Sl(n, \mathbb{K})$:

$$G = \left\{ \begin{bmatrix} A & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{bmatrix} \text{ for } A \in Sl(2, \mathbb{K}) \text{ and } x \in \mathbb{K}^2 \right\}$$

$$N = \left\{ \begin{bmatrix} I_2 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{bmatrix} \text{ for } x \in \mathbb{K}^2 \right\}$$

- Does G have property (T)?
- Assuming that pair (G, N) has property (T) and using using characteristic subgroups of $Sl(n, \mathbb{K})$ prove that it has property (T).

  Consider the embedding of $Sl(2, \mathbb{K}) \hookrightarrow Sl(n, \mathbb{K})$ given by $\begin{bmatrix} * & 0 & * & 0 \\ 0 & 1 & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & 0 & I_{n-3} \end{bmatrix}$.
- Prove that (G, N) has property (T) (this is hard)