Introduction

Banach algebra:

- algebra over $\mathbb{C}$;
- Banach space
- structures are compatible: $\|a+b\| \leq \|a\| + \|b\|$, $\|\lambda a\| = |\lambda|\|a\|$; $\|ab\| \leq \|a\|\|b\|$.

Banach $*$-algebra: Banach algebra with an involution $*$ ($*^2 = id$), which is isometric ($\|a^*\| = \|a\|$) anti-isomorphism (antilinear and $(ab)^* = b^* a^*$).

C^*-property: $\|a^* a\| = \|a\|^2$.

C^*-algebra is a Banach $*$-algebra satisfying the C^*-property.

Problem: In a C^*-algebra, $1 + a^* a$ is invertible for any a.

Examples: C^*-algebras

- algebra $C(X)$ of all continuous complex-valued functions on a locally compact Hausdorff space X;
- algebra $\mathbb{B}(H)$ of all bounded operators on a Hilbert space H;

Not C^*-algebras

- algebra $C^1[0,1]$ of functions with continuous derivative;
- algebra $C(0,1)$ of continuous functions;
- algebra $l^1(G)$ for a countable discrete group G (multiplication is given by the convolution).

Unitalization

We do not assume C^*-algebras to have a unit element, but if they have it then $1^* = 1$ and $\|1\| = 1$.

Theorem

Any non-unital C^*-algebra A is contained in a unital C^*-algebra A^+ as a maximal ideal of codimension 1.

Set $A^+ = A \oplus \mathbb{C}$;
$(a, \lambda)(b\mu) = (ab + \lambda b + \mu a, \lambda\mu)$; $(a, \lambda)^* = (a^*, \overline{\lambda})$,
$\|(a, \lambda)\| = \sup_{\|b\| \leq 1} \|ab + \lambda b\|$.
Unit: $(0, 1)$. Involution: $(a, \lambda)^* = (a^*, \overline{\lambda})$ (check that $*$ is isometry on A^+). Norm is the operator norm. Let $\mathbb{B}(A)$ be the algebra of bounded operators on A, $L_a : A \to A$ given by $L_a(b) = ab$. Then $A^+ \cong \{L_a + \lambda \cdot 1 : a \in A, \lambda \in \mathbb{C}\}$ (the norm on A^+ is taken from that on $\mathbb{B}(A)$; check that $\|a\| = \|L_a\|$).

C^*-property for A^+:

$\|(a,\lambda)\|^2 = \sup_{\|b\|\leq 1} \|ab+\lambda b\|^2$

$= \sup_{\|b\|\leq 1} \|b^*a^*ab + \lambda b^*a^*b + \overline{\lambda} b^*ab + |\lambda|^2 b^*b\|$

$\leq \sup_{\|b\|\leq 1} \|a^*ab + \lambda a^*b + \overline{\lambda} ab + |\lambda|^2 b\|$

$= \|(a^*a + \lambda a^* + \overline{\lambda} a, |\lambda|^2)\| = \|(a,\lambda)^*(a,\lambda)\|.$

The opposite inequality is easier:

$\|(a,\lambda)^*(a,\lambda)\| \leq \|(a,\lambda)^*\| \cdot \|(a,\lambda)\| = \|(a,\lambda)\|^2.$ □

Spectrum

Let A be a unital Banach algebra, $a \in A$.

Spectrum: $\mathrm{Sp}(a) = \{\lambda \in \mathbb{C} : \lambda \cdot 1 - a \text{ is not invertible}\}$.
Resolvent set: $\mathrm{Res}(a) = \mathbb{C} \setminus \mathrm{Sp}(a)$.
Resolvent function: $R_a(\lambda) = (\lambda \cdot 1 - a)^{-1}$

Theorem.

- $\mathrm{Sp}(a) \neq \emptyset$;
- $\mathrm{Sp}(a)$ is a compact set;
- if $\lambda \in \mathrm{Sp}(a)$ then $|\lambda| \leq \|a\|$;
- R_a is analytic on $\mathrm{Res}(a)$;
- if p is a polynomial then $\mathrm{Sp}(p(a)) = p(\mathrm{Sp}(a))$.

Let $a \in A$. Define its spectral radius by $r(a) = \sup_{\lambda \in \mathrm{Sp}(a)} |\lambda|$.

Theorem. $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} \le \|a\|$;
R_a is analytic on $\{\lambda \in \mathbb{C} : |\lambda| > r(a)\}$.

Theorem. If A is a simple unital abelian Banach algebra then $A \cong \mathbb{C}$.

Proof. Let $a \in A$, $a \neq \lambda \cdot 1$, $\alpha \in \mathrm{Sp}(a)$. Then $I = (a - \alpha \cdot 1)A$ is a (possibly non-closed) ideal. Any $x \in I$ is of the form $x = (a - \alpha \cdot 1)b$, $b \in A$, and cannot be invertible, since $a - \alpha \cdot 1$ is not invertible.
If $\|1 - x\| < 1$ then x is invertible (use $x^{-1} = 1 + (1 - x) + (1 - x)^2 + \dots$). So, if $x \in I$ then $1 \notin I$.
Moreover, if $x \in \bar{I}$ then $\mathrm{dist}(1, \bar{I}) \ge 1$, hence $1 \notin \bar{I}$, so $\bar{I} \neq A$ — contradiction with simplicity. □

For a commutative Banach algebra A we write M_A for the set of all non-zero multiplicative linear functionals (characters) of A.

Theorem. If $\varphi \in M_A$ then φ is continuous of norm 1.

Proof. Suppose there exists $a \in A$ such that $\|a\| < 1$, $|\varphi(a)| = 1$. Multiplying by λ, $|\lambda| = 1$, we may assume that $\varphi(a) = 1$. The series $b = \sum_{n\geq 0} a^n$ converges, and $a + ab = b$. Then $1 + \varphi(b) = \varphi(b) + \varphi(a)\varphi(b) = \varphi(b)$ — contradiction. Hence $\|\varphi\| \leq 1$. But $\varphi(1) = 1$. □

Theorem. There is a one-to-one correspondence between M_A and the set of maximal ideals of A given by $\varphi \mapsto \operatorname{Ker} \varphi$.

Injectivity: φ is determined by its kernel, which has codimension 1, and by $\varphi(1) = 1$.
Surjectivity: If $M \subset A$ is a maximal ideal then $\operatorname{dist}(1, M) \geq 1$, hence $1 \notin M$, so $\overline{M} \neq A$, hence $\overline{M} = M$. The quotient A/M is a simple unital abelian Banach algebra, hence $A/M \cong \mathbb{C}$. The quotient homomorphism is a character. □

M_A identified with the set of maximal ideal can be topologized. Let A' be the dual Banach space for A, $\mathbb{B}_1 9A')$ its unit ball. Then $M_A \subset \mathbb{B}_1(A')$.
The weak* topology on $\mathbb{B}_1(A')$ is determined by the pre-base $U_{a\varepsilon}(\varphi) = \{\psi \in \mathbb{B}_1(A') : |(\psi(a) - \varphi(a))| < \varepsilon\}$, $a \in A$, $\varepsilon \in (0, 2]$.
M_A is obviously Hausdorff: if $\psi \neq \varphi$ then there exists $a \in A$ such that $\psi(a) \neq \varphi(a)$.
By Banach–Alaoglu Theorem, $\mathbb{B}_1(A')$ is compact. If $\varphi_\alpha \to \varphi$, $\varphi_\alpha \in M_A$, then φ is also multiplicative, hence M_A is a closed subset of a compact space.

For a unital Banach algebra A, M_A is a compact Hausdorff space.

Non-unital case:
If A is a non-unital commutative Banach algebra then A is a maximal ideal in its unitalization A^+ corresponding to the character φ_0 given by $\varphi_0(a + \lambda \cdot 1) = \lambda$. For any other maximal ideal $M \subset A^+$ the set $M \cap A$ is an ideal of A of codimension 1. Then the quotient map $\varphi : A \to A/M \cap A$ is a character, and $\tilde{\varphi}(a + \lambda \cdot 1) = \varphi(a) + \lambda$ gives a character on A^+, which is a unique extension of φ. Then there is a one-to-one correspondence between M_A and $M_{A^+} \setminus \{\varphi_0\}$, hence M_A is locally compact, and M_{A^+} is its one-point compactification.

Gelfand Transform

For a commutative Banach algebra A, $a \in A$, $\varphi \in M_A$, the formula $\hat{a}(\varphi) = \varphi(a)$ defines a continuous function on M_A. Set $\Gamma(a) = \hat{a}$. This gives a Gelfand transform

$$\Gamma : A \to C_0(M_A).$$

Theorem. (1) Γ is a contractive algebra homomorphism; (2) its image separates points in M_A.

Note that if A is not unital then $\hat{a}(\varphi_0) = \varphi_0(a) = 0$ for any $a \in A$ hence $\hat{a} \in C_0(M_A)$.

Proof. (1) $\|\Gamma(a)\| = \|\hat{a}\| = \sup_{\varphi \in M_A} |\varphi(a)| \leq \|\varphi\| \cdot \|a\| \leq \|a\|$.
(2) If $\psi \neq \varphi$ then there exists $a \in A$ such that $\psi(a) \neq \varphi(a)$. □

Corollary. Let A be a unital commutative Banach algebra, $a \in A$. Then TFAE:

- a is invertible;
- $\hat{a}$ is invertible;
- $\hat{a}(\varphi) \neq 0$ for any $\varphi \in M_A$.

Thus $\mathrm{Sp}(a) = \mathrm{Sp}(\hat{a}) = \{\varphi(a) : \varphi \in M_A\}$, and $\|\hat{a}\| = r(a)$.

Proof. If a is invertible then $\Gamma(a)$ is invertible, and $\Gamma(a)^{-1} = \Gamma(a^{-1})$. If a is not invertible then consider the ideal $I = \overline{aA}$. It consists of non-invertibles (with distance from 1 bounded from below by 1), so $I \neq A$. Hence there exists a maximal ideal M such that $I \subset M \subset A$. Let $\varphi \in M_A$ be the corresponding character. Then $M = \mathrm{Ker}\,\varphi$, so $\hat{a}(\varphi) = \varphi(a) = 0$, hence $\hat{a}$ is not invertible.
As the range of $\hat{a} = \mathrm{Sp}(a)$, so $\|\hat{a}\| = r(a)$. □

Now let A be a unital C^*-algebra (not necessarily commutative).

Lemma. Let φ be a multiplicative functional on A. Then $\varphi(a^*) = \overline{\varphi(a)}$.

Proof. First, let $a^* = a$. The series $u_t = \sum_{n=0}^{\infty} \frac{(ita)^n}{n!}$ and $u_t^* = \sum_{n=0}^{\infty} \frac{(-ita)^n}{n!}$ converge, and $u_t^* = u_t^{-1}$ for any $t \in \mathbb{R}$. By the C^*-property, $\|u_t\|^2 = \|u_t^* u_t\| = \|1\| = 1$.

Then $|\varphi(u_t)| \leq 1$. But $\varphi(u_t) = \sum_{n=0}^{\infty} \frac{(it\varphi(a))^n}{n!} = e^{it\varphi(a)}$. Hence $e^{-t \operatorname{Im} \varphi(a)} = |e^{it\varphi(a)}| \leq 1$ for any $t \in \mathbb{R}$. So $\operatorname{Im} \varphi(a) = 0$.

Now let a be arbitrary. It decomposes as $a = b + ic$, where b, c are selfadjoint, and $a^* = b - ic$. As $\varphi(b)$, $\varphi(c)$ are real, the conclusion follows. □

Theorem. Let A be a commutative C^*-algebra. Then Γ is an isometric $*$-isomorphism $A \to C_0(M_A)$.

Proof. If $a^* = a$ then $\|a\|^2 = \|a^2\|$, hence $\|\Gamma(a)\| = \|\hat{a}\| = r(a) = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \lim_{n\to\infty}(\|a\|^{2^n})^{1/2^n} = \|a\|$.

If a is arbitrary, then $\|a\|^2 = \|a^* a\| = \|\Gamma(a^* a)\| = \|\Gamma(a)\|^2$.

Surjectivity of Γ follows from the Stone–Weierstrass Theorem ($\Gamma(A)$ separates points in M_A). □

Continuous functional calculus for normals

Let A be a C^*-algebra, $a \in A$ normal, i.e. $a^*a = aa^*$, and let $C^*(a) \subset A$ be the C^*-algebra generated by a (equivalently, the minimal C^*-subalgebra containing a).

Corollary. If $C^*(a)$ is unital then the Gelfand transform $C^*(a) \to C(\mathrm{Sp}(a))$ is an isometric $*$-isomorphism. If $C^*(a)$ is not unital then the image of the Gelfand transform is $C_0(\mathrm{Sp}(a) \setminus \{0\})$.

Indeed, $C^*(a)$ is commutative, so $C^*(a) \cong C(X)$ for some X (with necessary corrections in the non-unital case). Any its character φ is determined by $\varphi(a) = \lambda$. λ can be the value of φ iff λ lies in the range of the function $\hat{a}$, which is $\mathrm{Sp}(a)$. □

Continuous functional calculus for normals

Let $f \in C(\mathrm{Sp}(a))$. Then $f(a) := \Gamma^{-1}(f) \in C^*(a)$ (or $\in C^*(a)^+$, if the latter was not unital; if $0 \in \mathrm{Sp}(a)$ and $f(0) = 0$ then $f(a) \in C^*(a)$).

As Gelfand transform is an isometry, so if $f = \lim_{n\to\infty} p_n$, where $p_n = p_n(a, a^*)$ are polynomials, then $f(a) = \lim_{n\to\infty} p_n(a, a^*)$.

If $a \in A$ is normal then (1) $\mathrm{Sp}(f(a)) = f(\mathrm{Sp}(a))$;
(2) $g(f(a)) = (g \circ f)(a)$ for any function g continuous on $f(\mathrm{Sp}(a))$.

Proof: approximation by polynomials. □

Corollary. In a C^*-algebra A one has

- if $a \in A$ is normal then $\|a\| = r(a)$;
- if $a^* = a$ then $\mathrm{Sp}(a) \subset \mathbb{R} \subset \mathbb{C}$;
- if $a^*a = aa^* = 1$ then $\mathrm{Sp}(a) \subset \mathbb{S}^1 \subset \mathbb{C}$.

Positivity

Definition. An element a of a C^*-algebra is positive if $a^* = a$ and $\mathrm{Sp}(a) \subset [0, \infty)$. Notation: $a \geq 0$.

Lemma. For any positive $a \in A$ there exists a unique positive $b \in A$ such that $b^2 = a$.

Proof: The function $f(t) = \sqrt{t}$ is continuous on $[0, \infty)$, so put $b = f(a)$. If $c^2 = a$ then $c = f(c^2) = f(a) = b$. □

Lemma. Let $a^* = a$. There exist $a_+, a_- \geq 0$ such that $a = a_+ - a_-$ and $a_+ a_- = 0$.

Proof: Let $f_+(t) = \begin{cases} t, & \text{if } t \geq 0 \\ 0, & \text{if } t \leq 0 \end{cases}$, $f_-(t) = \begin{cases} 0, & \text{if } t \geq 0 \\ -t, & \text{if } t \leq 0 \end{cases}$, $a_\pm = f_\pm(a)$. □

Positivity

Lemma. Let $a^* = a$. TFAE:

1. $a \geq 0$;
2. $a = b^2$ for some selfadjoint b;
3. $\|\lambda \cdot 1 - a\| \leq \lambda$ for any $\lambda \geq \|a\|$;
4. $\|\lambda \cdot 1 - a\| \leq \lambda$ for some $\lambda \geq \|a\|$.

Proof. $1 \Rightarrow 2$ was already done, $3 \Rightarrow 4$ is trivial. Remaining implications $2 \Rightarrow 3$ and $4 \Rightarrow 1$ can be checked inside $C^*(a) \cong C(\mathrm{Sp}(a))$, where a corresponds to the function $f(t) = t$ (hence $\lambda \cdot 1 - a$ corresponds to the function $\lambda - t$ on $\mathrm{Sp}(a)$), positivity for a function f means that f takes only non-negative values, and the norm is the sup-norm. In particular, if $\sup_{t \in \mathrm{Sp}(a)} |\lambda - t| \leq \lambda$ for some $\lambda > \|a\|$ then $\mathrm{Sp}(a) \subset [0, \infty)$. □

Corollary. If $a, b \geq 0$ then $a + b \geq 0$.

Proof: Let $\lambda \geq \|a\|$, $\mu \geq \|b\|$, then $\lambda + \mu \geq \|a + b\|$, and $\|(\lambda + \mu)1 - (a + b)\| \leq \|\lambda \cdot 1 - a\| + \|\mu \cdot 1 - b\| \leq \lambda + \mu$. □

Thus the set of positive elements in a C^*-algebra is a cone.

Theorem. In a C^*-algebra, $a^*a \geq 0$ for any a.

Proof uses the following algebraic lemma:

Lemma. Let R be a ring, $a, b \in R$. $1 - ab$ is invertible iff $1 - ba$ is invertible.

Proof: if $c = (1 - ab)^{-1}$ then $d = (1 - ba)^{-1}$, where $d = 1 + bca$.

Corollary. $\mathrm{Sp}(ab) \cup \{0\} = \mathrm{Sp}(ba) \cup \{0\}$.

Proof: If $\lambda \cdot 1 - ab$ is (not) invertible and if $\lambda \neq 0$ then $1 - \frac{1}{\lambda}ab$ is (not) invertible. □

Positivity

Proof of the Theorem.

Set $b = a^*a$ (it is patently selfadjoint) and decompose it as $b = b_+ - b_-$, where $b_+, b_- \geq 0$, $b_+b_- = 0$. Set $c = b_-^{1/2}$, $t = ac$.

As the square root function is a limit of polynomials that vanish at 0, so $b_+b_- = 0$ implies $cb_+ = 0$.

Then $t^*t = ca^*ac = c(b_+ - b_-)c = -cb_-c = -b_-^2$, hence $-t^*t = b_-^2 \geq 0$.

Let $t = x + iy$ with x, y selfadjoint. Then $t^*t + tt^* = (x+iy)^*(x+iy) + (x+iy)(x+iy)^* = 2(x^2 + y^2) \geq 0$. As both $-t^*t$ and $t^*t + tt^*$ are positive, their sum is positive: $tt^* \geq 0$. Thus, $\mathrm{Sp}(tt^*) \subset [0, \infty)$, and $\mathrm{Sp}(t^*t) \subset (-\infty, 0]$, so $\mathrm{Sp}(t^*t) = \{0\}$. Finally, $\|b_-\|^2 = \|t^*t\| = r(t^*t) = 0$, so $b_- = 0$. □

Let a, b be selfadjoints. We write $a \leq b$ if $b - a \geq 0$.

Lemma. If $a \leq b$ then $x^* a x \leq x^* b x$ for any x.

Proof: Let $c \geq 0$, $c^2 = b - a$. Then
$x^*(b - a)x = x^* c^2 x = (cx)^*(cx) \geq 0$.

Lemma. Let $0 \leq a \leq b$, and let a be invertible. Then b is invertible and $b^{-1} \leq a^{-1}$.

Invertibility of a means that there exists $\varepsilon > 0$ such that $\mathrm{Sp}(a) \subset [\varepsilon, \infty)$. Then b enjoys the same property, hence is invertible.
Then $1 - b^{-1/2} a b^{-1/2} = b^{-1/2}(b - a) b^{-1/2} \geq 0$, which means that $(a^{1/2} b^{-1/2})^*(a^{1/2} b^{-1/2}) \leq 1$. As $\mathrm{Sp}(xy)$ and $\mathrm{Sp}(yx)$ coinside up to $\{0\}$, we have
$a^{1/2} b^{-1} a^{1/2} = (a^{1/2} b^{-1/2})(a^{1/2} b^{-1/2})^* \leq 1$. Multiplying by $a^{-1/2}$ both from the left and from the right, we get
$b^{-1} \leq a^{-1}$. □

In general, one has to be careful with the partial order $\leq$. Compare:

Theorem. If $0 \leq a \leq b$ implies that $0 \leq a^2 \leq b^2$ for any positive $a, b \in A$ then A is commutative.

Theorem. If $0 \leq a \leq b$ then $a^{1/2} \leq b^{1/2}$.

Approximate unit

Let A be a C^*-algebra. A set $\{e_\lambda : \lambda \in \Lambda\} \subset A$ is an approximate unit if

- Λ is a directed set;
- $e_\lambda \geq 0$ and $\|e_\lambda\| \leq 1$ for any $\lambda \in \Lambda$;
- $e_\lambda \leq e_\mu$ if $\lambda \leq \mu$;
- $\lim_{\lambda \in \Lambda} a e_\lambda = \lim_{\lambda \in \Lambda} e_\lambda a = a$.

If A is unital then Λ can contain one element, and 1 can be taken as e_λ.

Example 1. Let $A = C_0(0, 1]$ and let $\Lambda = [1, \infty)$. Let $f_\lambda(t) = \begin{cases} \lambda \cdot t, & \text{if } t \in (0, \frac{1}{\lambda}; \\ 1, & \text{if } t \in [\frac{1}{\lambda}, 1]. \end{cases}$ Then $\{f_\lambda\}$ is an approximate unit.

Example 2. Let $\mathbb{K}(H)$ be the algebra of compact operators on a separable Hilbert space H, with a fixed orthonormal basis $\{x_n\}_{n \in \mathbb{N}}$, and let $p_n \in \mathbb{K}(H)$ denote the projection onto the linear span of $x_1, \ldots, x_n$. Then $\{p_n\}$ is an approximate unit.

Approximate unit

Theorem. Any C^*-algebra A has an approximate unit. If A is separable then it has a countable approximate unit.

Proof. Set $\Lambda = \{a \in A : a \geq 0, \|a\| < 1\}$ ordered by $\leq$. We have to show that for any $a, b \in \Lambda$ we can find $c \in \Lambda$ such that $a \leq c$, $b \leq c$. $c = a + b$ is not a right choice. The idea is to re-scale spectra of a and b from $[0, 1]$ to $[0, \infty)$, then add the rescaled elements together, and then re-scale the sum back. For re-scaling forward, we use $f(t) = \frac{t}{1-t}$, and for re-scaling backward, we use $g(t) = \frac{t}{1+t} = 1 - \frac{1}{1+t}$, $g(f(t)) = t$.
Set $x = f(a)$, $y = f(b)$, $z = x + y$, $c = g(c)$. As $\mathrm{Sp}(z) \subset [0, N]$ for some $N \in \mathbb{R}$, so $\|c\| = \sup_{t \in \mathrm{Sp}(z)} g(t) \leq \frac{N}{N+1} < 1$, hence $c \in \Lambda$. As $1 + x \leq 1 + z$, so we have $(1 + x)^{-1} \geq (1 + z)^{-1}$, and $a = g(c) = 1 - (1 + x)^{-1} \leq 1 - (1 + z)^{-1} = c$. Symmetrically, $b \leq c$.

Proof (continued). Also, if $a, b \in \Lambda$, $a \leq b$, then $\|d - bd\|^2 = \|d^*(1-b)^2 d\| \leq \|d^*(1-b)d\| \leq \|d^*(1-a)d\|$ for any $d \in A$.

Take $d \in A$, $d \geq 0$. Set $a_n = g(nd) \in \Lambda$, $h_n(t) = t^2(1 - g(nt)) = \frac{t^2}{1+nt} \leq \frac{t}{n}$. Then $d - a_n d = h_n(d)$, hence $\|d - a_n d\| = \|h_n\|_{\mathrm{Sp}(d)} \to 0$ as $n \to \infty$.
Then $\lim_{b\in\Lambda} \|d - bd\|^2 \leq \lim_{n\to\infty}(\sup_{b\in\Lambda; b\geq a_n} \|d - bd\|^2) \leq \lim_{n\to\infty} \|d(1 - a_n)d\| = 0$.
If d is arbitrary, then
$\|d - bd\|^2 = \|(1-b)d^*d(1-b)\| \leq \|d^*d - bd^*d\| \to 0$.
Similarly, $\|d - db\| \to 0$ as $b \in \Lambda$.
The standard argument shows that if A is separable then one can find a subsequence in Λ with the required properties. □

Unless the converse is stated, by an ideal in a C^*-algebra we always mean a closed, two-sided ideal.

Lemma. If $J \subset A$ is an ideal then $J^* = J$.

Proof. Set $B = J \cap J^*$. Then B is a C^*-algebra, and $JJ^* \subset B$. Let $\{e_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit for B, and let $j \in J$. Then $\lim_{\lambda\in\Lambda} \|j^* - j^* e_\lambda\|^2 = \lim_{\lambda\in\Lambda} \|jj^* - jj^* e_\lambda - e_\lambda(jj^* - jj^* e_\lambda)\| \leq 2\lim_{\lambda\in\Lambda} \|jj^* - jj^* e_\lambda\| = 0$. As J is closed and $j^* e_\lambda \in J$ then $j^* \in J$. □

Unlike general rings, for C^*-algebras one has

Lemma. Let $J \subset A$ be an ideal in A, and let $I \subset J$ be an ideal in J. Then I is an ideal in A.

Let $i \in I$, $a \in A$. We need to show that $ia \in I$. Assume first that i is positive. Then $i = (i^{1/2})^2$, hence $i^{1/2}a \in J$ and $i^{1/2} \cdot i^{1/2}a \in I$. The case of arbitrary $i \in I$ can be done by decomposing i as a linear combination of 4 positive elements.

Let $J \subset A$ be an ideal. Then (J is closed) the quotient Banach algebra A/J is well defined with the norm given by $\|\dot{a}\| = \inf_{j \in J} \|a - j\|$, where $\dot{a} = a + J \in A/J$. As J is symmetric, so $\|\dot{a}^*\| = \|\dot{a}\|$.

Theorem. A/J is a C^*-algebra.

Proof. We need to check the C^*-property. First, we claim that $\|\dot{a}\| = \lim_{\lambda \in \Lambda} \|a - ae_\lambda\|$ for any approximate unit $\{e_\lambda\}$ in J. As $ae_\lambda \in J$, so we obviously have $\|\dot{a}\| \leq \|a - ae_\lambda\|$.
For any $\varepsilon > 0$ there exists $j \in J$ such that $\|a - j\| < \|\dot{a}\| + \varepsilon$.
Then $\lim_{\lambda \in \Lambda} \|a - ae_\lambda\| \leq \lim_{\lambda \in \Lambda} \|(a - j)(1 - e_\lambda)\| + \|j - je_\lambda\| \leq \|a - j\| < \|\dot{a}\| + \varepsilon$. As ε is arbitrary, the claim is proved.
$\|\dot{a}^* \dot{a}\| \leq \|\dot{a}^*\| \cdot \|\dot{a}\| = \|\dot{a}\|^2$, so itremains to check the opposite inequality:
$\|\dot{a}\|^2 = \lim_{\lambda \in \Lambda} \|a(1 - e_\lambda)\|^2 = \lim_{\lambda \in \Lambda} \|(1 - e_\lambda)a^* a(1 - e_\lambda)\| \leq \lim_{\lambda \in \Lambda} \|a^* a(1 - e_\lambda)\| = \|\dot{a}^* \dot{a}\|$. □

$*$-homomorphisms

Theorem. Let A, B be C^*-algebras, $\pi : A \to B$ a non-zero $*$-homomorphism. Then $\|\pi\| = 1$ and $\pi(A)$ is a C^*-subalgebra in B. If π is injective then π is an isometry. In general, π factorizes as $\pi = \dot{\pi} \circ q$, where $q : A \to A/\operatorname{Ker}\pi$, and $\dot{\pi}$ induced by π is an isometric $*$-isomorphism of $A/\operatorname{Ker}\pi$ onto $\pi(A)$.

Proof. We prove this for the case when A, B are unital, and $\pi(1) = 1$. (The general proof is slightly more technical.) If $a \in A$ is invertible then $\pi(a)$ is invertible, hence $\mathrm{Sp}_B(\pi(a)) \subset \mathrm{Sp}_A(a)$ for any $a \in A$, therefore, $r(\pi(a)) \le r(a)$. Then
$\|\pi(a)\|^2 = \|\pi(a^*a)\| = r(\pi(a^*a)) \le r(a^*a) = \|a^*a\| = \|a\|^2$,
hence $\|\pi\| \le 1$. As $\pi(1) = 1$, $\|\pi\| = 1$.
Therefore, π is continuous, $\operatorname{Ker}\pi$ is a closed ideal in A, and π factorizes through $A/\operatorname{Ker}\pi$ with $\dot{\pi}$ injective.

Proof (continued). Suppose that $\dot{\pi}$ is not an isometry. Then there exists $a \in A$ such that $\|\dot{\pi}(a)\| < \|a\|$. Let $r = \|\dot{\pi}(a^*a)\| < \|a^*a\| = s$, and let $f \in C[0, s]$ be the function such that $f(t) = 0$ for $t \in [0, r]$, and $f(s) = 1$. Then $\dot{\pi}(f(a^*a)) = f(\dot{\pi}(a^*a)) = 0$, but as $f \neq 0$ on $\mathrm{Sp}(a^*a)$, so $f(a^*a) \neq 0$ — a contradiction with injectivity of $\dot{\pi}$. So, $\dot{\pi}$ is an isometry, and $\pi(A)$ is closed. □

Corollary. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on A, both making A a C^*-algebra, then $\|\cdot\|_1 = \|\cdot\|_2$.

Remark. It is hidden here that A is closed with respect to both norms. If not, there may be many C^*-norms on a $*$-algebra

States

A linear functional $\varphi : A \to \mathbb{C}$ is called positive if $\varphi(a) \geq 0$ for any positive $a \in A$. A positive linear functional φ is called a state if $\|\varphi\| = 1$.

Example 1. Let $A = C(X)$ for a compact Hausdorff X, and let $x_0 \in X$. The evaluation map $\varphi(f) = f(x_0)$, $f \in C(X)$, is a state.

Example 2. Let $\mathbb{B}(H)$ be the algebra of linear operators on a Hilbert space H, and let $\xi \in H$. Then the map $\varphi(a) = \langle a\xi, \xi \rangle$, $a \in \mathbb{B}(H)$, is a positive linear functional. It is a state iff $\|\xi\| = 1$.

If φ is a positive linear functional then the formula $\langle a, b \rangle = \varphi(b^* a)$defines a positive semi-definite sesquilinear form on A. An immediate corollary is the Cauchy–Schwarz inequality $|\varphi(b^* a)|^2 \leq \varphi(a^* a)\varphi(b^* b)$.

Lemma. Let φ be a positive linear functional on a C^*-algebra A. Then $\|\varphi\| = \lim_{\lambda\in\Lambda} \varphi(e_\lambda)$ for any approximate unit $\{e_\lambda\}$ in A. In particular, if A is unital then $\|\varphi\| = \varphi(1)$.

Proof. We prove this Lemma under the additional assumption that φ is bounded. Under this assumption, the set $\varphi(e_\lambda)$ is increasing and bounded, hence converges to some $M = \lim_{\lambda\in\Lambda} \varphi(e_\lambda)$. Obviously, $\|\varphi\| \geq M$. To prove the opposite inequality, take $\varepsilon > 0$ and find $a \in A$ such that $\|a\| \leq 1$ and $|\varphi(a)| > \|\varphi\| - \varepsilon$. Then $|\varphi(a)|^2 = \lim_{\lambda\in\Lambda} |\varphi(e_\lambda a)|^2 \leq \lim_{\lambda\in\Lambda} \varphi(e_\lambda^2)\varphi(a^* a) \leq M\|\varphi\|$ (we used here $e_\lambda^2 \leq e_\lambda$). Thus we have $(\|\varphi\| - \varepsilon)^2 \leq M\|\varphi\|$. As ε is arbitrary, so $\|\varphi\|^2 \leq M\|\varphi\|$, $\|\varphi\| \leq M$. The unital case follows. □

GNS construction

Gelfand–Naimark–Segal construction

Theorem. Let φ be a positive linear functional on a C^*-algebra A. Then there exist a Hilbert space H_φ, a representation π_φ of A on H_φ, and a vector $\xi_\varphi \in H_\varphi$, cyclic for π_φ, such that $\|\xi_\varphi\|^2 = \|\varphi\|$ and $\quad \varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle, \quad a \in A.$

Proof. Set $N = \{a \in A : \varphi(a^*a) = 0\}$,
$N' = \{a \in A : \varphi(b^*a) = 0 \forall b \in A\}$. Both are closed. Due to the Cauchy–Schwarz inequality $|\varphi(b^*a)|^2 \leq \varphi(a^*a)\varphi(b^*b)$, $N' = N$, hence N is a linear subspace in A. Moreover, N is a left ideal: if $n \in N$, $a, b \in A$ then $\varphi(b^*(an)) = \varphi((a^*b)^*n) = 0$, hence $an \in N$.

Then we can pass to the quotient Banach space A/N and define a positive definite sesquilinear form $\langle \dot{x}, \dot{y} \rangle = \varphi(y^*x)$, where $\dot{x}$ denotes the class $x + N \in A/N$ of $x \in A$. (It is well-defined and if $\langle \dot{x}, \dot{x} \rangle = 0$ then $\dot{x} = 0$.)

Proof (continued). The inner product $\langle \dot{x}, \dot{y} \rangle$ defines a norm on A/N. Let H_φ denote the closure of A/N with respect to this norm. To define π_φ, let us first define a representation π_0 of A on A/N by $\pi_0(a)\dot{x} = (ax)^{\cdot}$ (well-defined because N is a left ideal). π_0 is obviously linear and $\pi_0(ab) = \pi_0(a)\pi_0(b)$ for any $a, b \in A$. It is also symmetric: $\langle \pi_0(a^*)^*\dot{x}, \dot{y} \rangle = \langle \dot{x}, \pi_0(a^*)\dot{y} \rangle = \langle \dot{x}, (a^*y)^{\cdot} \rangle = \varphi((a^*y)^*x) = \varphi(y^*ax) = \langle \pi_o(a)\dot{x}, \dot{y} \rangle$, hence $\pi_0(a^*) = \pi_0(a)^*$. Continuity of π_0 follows from the estimate $\|\pi_0(a)\|^2 = \sup_{\|\dot{x}\| \leq 1} \|\pi_0(a)\dot{x}\|^2 = \sup_{\|\dot{x}\| \leq 1} \varphi(x^*a^*ax) \leq \sup_{\|\dot{x}\| \leq 1} \varphi(x^*x)\|a^*a\| = \sup_{\|\dot{x}\| \leq 1} \|\dot{x}\|^2 \cdot \|a\|^2$ (we use here $a^*a \leq \|a^*a\| \cdot 1$). By continuity, π_0 extends to a representation π_φ on the completion H_φ of A/N.

If A is unital then set $\xi_\varphi = \dot{1}$. It is cyclic: $\pi_\varphi(A)\xi_\varphi = A/N$ is dense in H_φ. Also $|\varphi\| = \varphi(1) = \|\xi_\varphi\|^2$. Finally, $\langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle = \varphi(1^*a1) = \varphi(a)$.
If A is not unital then the proof is more technical: one has to prove that, for an approximate unit $\{e_\lambda\}$ the limit $\lim_{\lambda \in \Lambda} \dot{e}_\lambda$ exists, and then set $\xi_\varphi = \lim_{\lambda \in \Lambda} \dot{e}_\lambda$. We skip the details. □

Lemma. Let φ be a linear functional on a C^*-algebra A such that $\|\varphi\| = 1 = \lim_{\lambda\in\Lambda} \varphi(e_\lambda)$ for some approximate unit $\{e_\lambda\}$. Then φ is a state.

Proof. First, we reduct this to the unital case. Let $\tilde{\varphi} : A^+ \to \mathbb{C}$ be a Hahn–Banach extension of φ to A^+, and let $\tilde{\varphi}(1) = \alpha \in \mathbb{C}$. As $\|\tilde{\varphi}\| = \|\varphi\| = 1$, so $|\alpha| \leq 1$. As $\|2e_\lambda - 1\| \leq 1$, so $|\tilde{\varphi}(2e_\lambda - 1)| \leq 1$. Passing to the limit, we get $|2 - \alpha| \leq 1$. Hence $\alpha = 1$. Now we may assume that A is unital, and that $\|\varphi\| = 1 = \varphi(1)$.

Claim. If $a \in A$ is selfadjoint then $\varphi(a) \in \mathbb{R}$.

The proof follows from the claim. Let $0 \leq a \leq 1$. Then $\|2a - 1\| \leq 1$, hence $|\varphi(2a - 1)| \leq 1$. As $\varphi(a)$ is real, so we get $-1 \leq 2\varphi(a) - 1 \leq 1$, hence $\varphi(a) \geq 0$. Positivity is proved (modulo the Claim). □

Proof of the Claim. Let $a^* = a$, $\|a\| = 1$. Take $n \in \mathbb{N}$.
$\|a + in1\|^2 = \|(a - in1)(a + in1)\| = \|a^2 + n^2 1\| = n^2 + 1$.
Similarly, $\|a - in1\|^2 = n^2 + 1$.
Then $|\varphi(a \pm in1)| = |\varphi(a) \pm in| \leq \sqrt{n^2 + 1}$. As this holds for any $n \in \mathbb{N}$, so $\varphi(a) \in [-1, 1]$, hence it is real. □

Lemma. Let $B \subset A$ be a unital C^*-subalgebra of a unital C^*-algebra A (with common unit), φ a state on B. Then there exists a state $\tilde{\varphi}$ on A, that extends φ.

Proof. Let $\tilde{\varphi}$ be a Hahn–Banach extension of φ. As $\|\tilde{\varphi}\| = 1 = \tilde{\varphi}(1)$, so $\tilde{\varphi}$ is a state. □

Corollary. Let $a \in A$ be selfadjoint. Then there exists a state φ on A such that $|\varphi(a)| = \|a\|$.

Proof. If A is unital then set $B = C^*(a)^+ \subset A$. Either $r(a)$ or $-r(a)$ belongs to $\mathrm{Sp}(a)$, and one of the two formulas $\varphi(f) = f(\pm r(a))$, $f \in C(\mathrm{Sp}(a))$, gives a required state. □

Theorem (Gelfand–Naimark). Any C^*-algebra A is isometrically $*$-isomorphic to a closed $*$-subalgebra of $\mathbb{B}(H)$ for some Hilbert space H. If A is separable then H can be taken separable.

Proof. Let $S(A)$ denote the set of all states on A. For any $\varphi \in S(A)$, apply the GNS construction. Then $\pi = \oplus_{\varphi \in S(A)} \pi_\varphi$ is a representation of A on the Hilbert space $\oplus_{\varphi \in S(A)} H_\varphi$.

Take $a \in A$. Let $\varphi \in S(A)$ satisfy $\varphi(a^*a) = \|a^*a\|$. Then $\|\pi(a)\|^2 \geq \|\pi_\varphi(a)\|^2 \geq \|\pi_\varphi(a)\xi_\varphi\|^2 = \langle \pi_\varphi(a^*a)\xi_\varphi, \xi_\varphi \rangle = \varphi(a^*a) = \|a^*a\|$. On the other hand, π is a $*$-homomorphism, hence $\|\pi\| = 1$, therefore $\|\pi(a)\| = \|a\|$ for any $a \in A$.

If A is separable then one can use a dense sequence $\{a_n\}$ in A, and the corresponding sequence $\{\varphi_n\}$ of states such that $\varphi_n(a_n^* a_n) = \|a_n\|^2$. Set $\pi = \oplus_{n \in \mathbb{N}} \pi_{\varphi_n}$. □

Let A be a C^*-algebra, H a Hilbert space, $\pi : A \to \mathbb{B}(H)$ a representation. $\pi : A \to \mathbb{B}(H)$ is algebraically irreducible if is has no proper invariant subspace. It is topologically irreducible if it has no proper closed invariant subspace. For a set $S \subset \mathbb{B}(A)$ we denote by $A^!$ its commutant in $\mathbb{B}(H)$, i.e. $S^! = \{a \in \mathbb{B}(H) : as = sa \forall s \in S\}$.

TFAE:

- $\pi(A)^! = \mathbb{C} \cdot 1$;
- π is algebraically irreducible;
- π is topologically irreducible.

Proof requires Borel functional calculus in $\mathbb{B}(H)$, the weak topology on $\mathbb{B}(H)$, and is based on Kaplansky density Theorem.

Representation π is non-degenerate if $\pi(A)H = \{\pi(a)\xi : a \in A, \xi \in H\}$ is dense in H.

Irreducible $\Rightarrow$ non-degenerate: if $\overline{\pi(A)H}$ differs from H then it is an invariant subspace.

Let $J \subset A$ be an ideal, $\pi : J \to \mathbb{B}(H)$ a non-degenerate representation. Then π uniquely extends to a representation $\tilde{\pi}$ of A on H. $\tilde{\pi}$ is irreducible iff π is irreducible.

Proof. For $a \in A$, $j \in J$, $\xi \in H$, set $\tilde{\pi}(a)(\pi(j)\xi) := \pi(aj)\xi$. It may happen that $\pi(j_1)\xi_1 = \pi(a_2)\xi_2$. Let $\{e_\lambda\}$ be an approximate unit in J. Then $\pi(aj_1)\xi_1 = \lim_{\lambda\in\Lambda} \pi(ae_\lambda j_1)\xi_1 = \lim_{\lambda\in\Lambda} \pi(ae_\lambda)\pi(j_1)\xi_1 = \lim_{\lambda\in\Lambda} \pi(ae_\lambda)\pi(j_2)\xi_2 = \pi(aj_2)\xi_2$, so $\tilde{\pi}$ is well defined (and uniquely defined) on the dense set $\pi(A)H$. It remains to extend it by continuity to H, and the estimate $\|\tilde{\pi}(a)(\pi(j)\xi)\| = \lim_{\lambda\in\Lambda} \|\pi(ae_\lambda)\| \cdot \|\pi(j)\xi\| \leq \|a\| \cdot \|\pi(j)\xi\|$ shows that $\tilde{\pi}$ is continuous.

If $\tilde{\pi}$ has an invariant subspace then π has it as well. Let us check the opposite. Let $L \subset H$ be a closed invariant subspace for π. Then $\overline{\tilde{\pi}(A)L} = \overline{\tilde{\pi}(A)\pi(J)L} = \overline{\pi(J)L} \subset L$. □

Let $\pi : A \to \mathbb{B}(H)$ be an irreducible representation, $J \subset A$ an ideal. Then $\pi|_J$ is irreducible.

Proof. Suppose the contrary: there exists a closed subspace $L \subset H$ invariant under $\pi(J)$, i.e. $\pi(J)L \subset L$. Then $\pi(A)\pi(J)L \subset L$. □

Finitedimensional C^*-algebras

Let A be a finitedimensional C^*-algebra. Then it is unital.

Proof. As A is finitedimensional, so the approximate unit has an accumulation point. Obviously, this is the unit element. □

Finitedimensional C^*-algebras

First, let us specify the Gelfand–Naimark Theorem.

For a finitedimensional C^*-algebra A there exists a faithful representation $\pi : A \to \mathbb{B}(H)$ on a finitedimensional Hilbert space H.

Proof. To show injectivity of $\pi = \oplus \pi_\varphi$, we don't need the exact equality $\|\pi_\varphi(a)\| = \|a\|$. It suffices to have the estimate $\|\pi_\varphi(a)\| > 1/2\|a\|$. For each $a \in A$ find an open set $U_a \subset S(A)$ of states satisfying this estimate. As A is finitedimensional, so $S(A)$ is compact, hence we can find a finite cover by the sets of the form U_a. Pick one state in each U_a and take a direct sum of GNS representations with these states. □

Now it is easy to decompose a faithful finitedimensional representation into a direct sum of irreducible ones. Let π be one of those, and let H be the corresponding finitedimensional Hilbert space.

Lemma. If π is irreducible and H finitedimensional then $\pi(A)$ contains a minimal projection of rank 1.

By projections we mean selfadjoint projections.
Proof. As $1 \in \pi(1)$ is a projection, so one can find a minimal non-zero projection $p = \pi(a)$ in $\pi(A)$ (minimal with respect to the patial order $\leq$). Consider the set $p\pi(A)p \subset \mathbb{B}(H)$. Suppose there is $b \in A$ such that $p\pi(b)p = \pi(aba)$ is not a scalar, i.e. has at least two eigenvalues, λ_1 and λ_2. Let $f \in C(\mathrm{Sp}(aba))$ satisfy $f(\lambda_1) = 1$, $f(\lambda) = 0$ for all other eigenvalues λ, and let $g = 1 - f$. Then $\pi(f(aba)) = f(\pi(aba))$ and $\pi(g(aba)) = g(\pi(aba))$ are projections with $\pi(f(aba)) + \pi(g(aba)) = \pi(a) = p$, both non-trivial. This contradicts to minimality of p. Thus, $p\pi(A)p$ consists only of scalars.

Suppose that $\mathrm{rk}\, p > 1$. Take two orthonormal vectors $\xi, \eta \in \mathrm{Im}\, p$. Then $\langle \pi(c)\xi, \eta \rangle = \langle p\pi(c)p\xi, \eta \rangle = \lambda \langle \xi, \eta \rangle = 0$ for any $c \in A$, where λ is some scalar, hence $\pi(A)\xi$ is an invariant space — contradiction with irreducibility.

Lemma. If π is irreducible and H finitedimensional then $\pi(A) = \mathbb{B}(H)$.

Proof. Let $\xi \in H$. Irreducibility implies that for any $\eta \in H$ there exists $c \in A$ such that $\pi(c)\xi = \eta$ (otherwise the linear subspace $\pi(A)\xi$ would be proper).

Now let $p = \pi(a)$ be a minimal projection of rank 1 in $\pi(A)$, and let ξ be a unit vector in the image of p. Then $p\omega = \langle \omega, \xi \rangle \xi$. If $\pi(c)\xi = \eta$ then $\pi(ca)\omega = \langle \omega, \xi \rangle \eta$. For $\zeta \in H$ let $b \in A$ satisfy $\pi(b)\zeta = \xi$. Then $\pi(cab)\omega = \langle \omega, \zeta \rangle \eta$. The operators of this form span $\mathbb{B}(H)$. □

Corollary. If A is a finitedimensional C^*-algebra then there exists a surjective $*$-homomorphism $\rho : A \to M_n$, where M_n is the matrix algebra $\mathbb{B}(H)$ of operators on an n-dimensional Hilbert space H.

Theorem. A finitedimensional C^*-algebra A is isometrically $*$-isomorphic to $M_{n_1} \oplus \cdots \oplus M_{n_k}$.

Proof. Let $\rho : A \to M_n$ for some n a surjective $*$-homomorphism, and let $e \in B = \operatorname{Ker} \rho$ be the unit of the ideal B.

If $a \in A$ then $a = eae + (1-e)ae + ea(1-e) + (1-e)a(1-e)$, with the first 3 summands being from B (where 1 is the unit of A). Note that as e is the unit of B, so $ea(1-e) = (1-e)ae = 0$ for any $a \in A$. Thus $A = eAe \oplus (1-e)A(1-e)$ (the sum is obviously a direct sum), where $eAe = B$. Define a $*$-homomorphism $\varphi : A \to B \oplus M_n$ by $\varphi(a) = (eae, \rho(a))$. As $\dim(1-e)A(1-e) = \dim M_n$, so it suffices to check injectivity of φ. If $\rho(a) = 0$ then $a \in B$, hence $a = eae$, which is zero. $\square$

Any two non-zero $*$-homomorphisms $\varphi, \psi : M_n \to M_n$ are unitarily equivalent, i.e. $\psi = \mathrm{Ad}_U \circ \varphi$ for some unitary $U \in M_n$.

If $m < n$ then any $*$-homomorphism $M_n \to M_m$ is a zero map.

If $n < m < 2n$ then there is no unital $*$-homomorphisms from M_n to M_m, but there is exactly one (up to unitary equivalence) $*$-homomorphism.

If $m = 2n$ then there are exactly two non-trivial $*$-homomorphisms $M_n \to M_m$: one is non-unital and takes rank 1 projections to rank 1 projections; another is unital and takes rank 1 projections to rank 2 projections.

If $kn \leq m < (k+1)n$ then there are exactly k non-trivial $*$-homomorphisms $M_n \to M_m$, taking rank 1 projections to rank k projections.

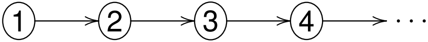
Thus any non-trivial $*$-homomorphism $M_n \to M_m$ is characterized by a single positive integer: the rank of the image of rank 1 projections. If it is trivial then the corresponding integer iz zero.

Let $A = M_{n_1} \oplus \cdots \oplus M_{n_k}$, $B = M_{m_1} \oplus \cdots \oplus M_{m_l}$ be two finitedimensional C^*-algebras, $\varphi : A \to B$ a $*$-homomorphism. Let $\alpha_i : M_{n_i} \to A$ and $\beta_j : B \to M_{m_j}$ be the obvious inclusion and projection maps respectively. Then $\beta_j \circ \varphi \circ \alpha_i : M_{n_i} \to M_{m_j}$ is characterized by an integer N_{ij}, and φ is characterized by the set of these numbers. This can be represented as a picture, where A is depicted by k points labeled by the dimensions $n_1, \ldots, n_k$, B is depicted by l points labeled by the dimensions $m_1, \ldots, m_l$, and each point labeled by n_i is connected to the point labeled by m_j by N_{ij} arrows.

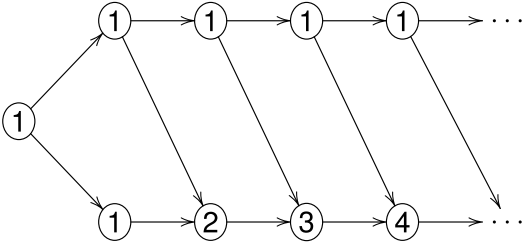
A C^*-algebra A is an AF algebra if it has C^*-subalgebras $A_1 \subset A_2 \subset \cdots \subset A$ such that $\cup_{n=1}^{\infty} A_n$ is dense in A.

Drawing each inclusion $A_n \to A_{n+1}$ by points and arrows allows to characterize AF algebras by diagrams, called Bratteli diagrams.

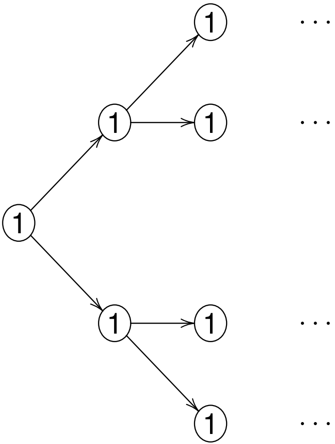
The algebra $\mathbb{K} = \mathbb{K}(H)$ of compact operators on a separable Hilbert space H is the closure of the union of matrix algebras $M_n = p_n\mathbb{K}p_n$, $n \in \mathbb{N}$, where p_n denotes the projection onto the linear span of the first n vectors of a fixed orthonormal basis of H, with the inclusion $M_n \subset M_{n+1}$ as the upper left corner. This corresponds to the Bratteli diagram



Its unitalization $\mathbb{K}^+$ is the closure of $\cup_{n=1}^{\infty} A_n$, where $A_n = \mathbb{C}p_n^{\perp} + p + n\mathbb{K}^+p_n \cong \mathbb{C} \oplus M_n$ with the Bratteli diagram

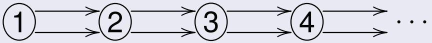


Let K be the Cantor set, $K = \cap_{n=0}^{\infty} X_n$, where X_n is the union of 2^n intervals of length $1/3^n$. Let $A_n \subset C(K)$ consists of functions that are locally constant on X_n. Then $A_n \cong \mathbb{C}^{2^n}$, and $\cup_{n=0}^{\infty} A_n$ is dense in $C(K)$. Here is the corresponding diagram:



CAR algebra

Let H be a separable Hilbert space with a fixed decomposition $H = H_1 \oplus H_2$ with $H_1 \cong H_2$. Let each of these two Hilbet subspaces is also a direct sum $H_i = H_{i,1} \oplus H_{i,2}$, etc. Let A_1 consist of scalar operators on H, A_2 consist of 2×2 matrices with scalar entries with respect to the decomposition $H_1 \oplus H_2$, etc. Then $A_n \cong M_{2^{n-1}}$. The inclusion $A_n \to A_{n+1}$ is given by $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$. The closure of $\cup_{n=1}^{\infty} A_n$ is a C^*-algebra called the CAR algebra with the Bratteli diagram



Lemma. If two AF algebras A and B have the same Bratteli diagrams then they are $*$-isomorphic.

Lemma. Linear combinations of projections are dense in any AF algebra.

Proof. Any element of a matrix algebra (hence, of a finitedimensional C^*-algebra) is a linear combination of projections. □

Corollary. $C[0,1]$ is not an AF algebra.

Caution: C^*-subalgebras of AF algebras need not be AF.

Let X be a compact metric space, then there exists a continuous surjection $K \to X$, which induces an injective $*$-homomorphism $C(X) \subset C(K)$.

Instead of matrix algebras, one may use other C^*-algebras as 'building blocks'.
Let $C([0,1]; M_m)$ be the C^*-algebra of all continuous functions on $[0,1]$ taking values in M_m. Even such C^*-algebras may have non-trivial subalgebras.

Example: Dimension-drop algebras.
Set $D_m = \{f \in C([0,1]; M_m) : f(0) = 0, f(1) \text{ is diagonal}\}$.

This is a C^*-algebra. If time allows, later we shall see that its K-groups are torsion.

A C^*-algebra is called an *AI* algebra if it contains C^*-subalgebras $A_1 \subset A_2 \subset \ldots \subset A$ such that each A_n is a C^*-subalgebra of $C([0,1]; M_m)$ for some m, and if $\cup_{n=1}^{\infty} A_n$ is dense in A.

For a C^*-algebra A, let $M_n(A)$ be the set of all $n \times n$ matrices with entries from A. Let $\pi : A \to \mathbb{B}(H)$ be a faithful $*$-representation on a Hilbert space H. Set $\widetilde{H} = H \oplus \cdots \oplus H$ (n times). Then there is a canonical $*$-isomorphism $M_n(\mathbb{B}(H)) \cong \mathbb{B}(\widetilde{H})$.
Define $\tilde{\pi} : M_n(A) \to \mathbb{B}(\widetilde{H})$ by $\tilde{\pi}((a_{ij})_{i,j=1}^n) = (\pi(a_{ij}))_{i,j=1}^n$. Then $\tilde{\pi}$ is a faithful $*$-representation of $M_n(A)$. Set $\|(a_{ij})_{i,j=1}^n\| = \|\tilde{\pi}((a_{ij})_{i,j=1}^n)\|$. Then $M_n(A)$ can be considered as a norm-closed $*$-subalgebra of $\mathbb{B}(\widetilde{H})$, hence it is a C^*-algebra. As all C^*-norms on a C^*-algebra are the same, the norm on $M_n(A)$ doesn't depend on choice of π (provided it is faithful).

Similarly, For a (locally) compact Hausdorff X, the set of all continuous A-valued functions on X (vanishing at infinity) is a C^*-algebra $C(X; A)$ with the norm $\|f\| = \sup_{x \in X} \|f(x)\|$.
This starts the theory of tensor products of C^*-algebras — a very interesting topic, which is beyond our short course. A good reference is the book by N. Brown and N. Ozawa.

Toeplitz algebra

Let $L^2(\mathbb{S})$ be the Hilbert space of square-integrable functions on the unit circle $\mathbb{S} \subset \mathbb{C}$, with the basis $\{e_n\}_{n \in \mathbb{Z}}$, $e_n = z^n$, $z \in \mathbb{C}$. Let $H^2(\mathbb{S}) \subset L^2(\mathbb{S})$ be the subspace spanned by $e_0, e_1, e_2, \ldots$, and let $L^\infty(\mathbb{S})$ be the C^*-algebra of essentially bounded measurable functions.

For $g \in L^\infty(\mathbb{S})$, set $M_g(f) = gf$, $f \in L^2(\mathbb{S})$. $\|M_g\| = \|g\|$. Define an operator T_g on $H^2(\mathbb{S})$ by $T_g h = PM_g h$, where P denotes the projection onto $H^2(\mathbb{S})$. In particular, $T_z e_n = e_{n+1}$, so T_z is the right shift on $H^2(\mathbb{S})$.

For an operator T on a Hilbert space H define its essential norm $\|T\|_{ess}$ as its distance to the set $\mathbb{K}(H)$ of compact operators (which is the norm on the quotient C^*-algebra $\mathbb{B}(H)/\mathbb{K}(H)$).

Lemma. Let $g \in L^\infty(\mathbb{S})$. Then $T_g^* = T_{\overline{g}}$, and $\|T_g\| = \|T_g\|_{ess} = \|g\|$.

Proof. The first statement is trivial. As regards the norms, one has, trivially, $\|T_g\|_{ess} \le \|T_g\| \le \|M_g\| = \|g\|$.

To prove the opposite estimate, take $\varepsilon > 0$ and find $p = \sum_{k=-N}^{N} a_k z^k \in L^2(\mathbb{S})$ such that $\|p\|_2 = 1$ and $\|gp\|_2 > \|g\| - \varepsilon$ (here $\|\cdot\|_2$ is the Hilbert space norm on $L^2(\mathbb{S})$).

Note that $z^n p \in H^2(\mathbb{S})$ for $n > N$. Let $gp = \sum_{k=-\infty}^{\infty} b_k e_e$. Then $T_g(z^n p) = P(z^n gp) = \sum_{k=-n}^{\infty} b_k e_k$, and there exists $N_0 > N$ such that $\|T_g(z^n p)\|_2 > \|gp\|_2 - \varepsilon$ for any $n \ge N_0$. Thus, for $n > N_0$ one has $\|T_g(z^n p)\| > \|g\| - 2\varepsilon$. Let $\xi_i = z^{3Ni} p \in H^2(\mathbb{S})$. These vectors have length 1 and are orthogonal to each other. Let $k \in \mathbb{K}(H)$. Then $\|T_g - k\| \ge \sup_{i\to\infty} \|(T_g - k)\xi_i\| = \limsup_{i\to\infty} \|T_g \xi_i\| \ge \|g\| - 2\varepsilon$, hence $\|T_g\|_{ess} \ge \|g\|$. □

Let $H^\infty(\mathbb{S}) = H^2(\mathbb{S}) \cap L^\infty(\mathbb{S})$.

Lemma. Let $g \in L^\infty(\mathbb{S})$, $h \in H^\infty(\mathbb{S})$. Then $H^2(\mathbb{S})$ isinvariant for M_h, and $T_g T_h = T_{gh}$, $T_{\overline{h}} T_g = T_{\overline{h}g}$.

Proof. If $f \in H^2(\mathbb{S})$ then $T_h f = P(hf) = hf \in H^2(\mathbb{S})$ (as both are series in z^n for $n \geq 0$).
$T_g T_h(f) = T_g(hf) = P(ghf) = T_{gh}(f)$;
$T_{\overline{h}} T_g = (T_{\overline{g}} T_h)^* = T^*_{\overline{g}h} = T_{\overline{h}g}$. □

Lemma. If $g \in L^\infty(\mathbb{S})$ then $T_z T_g - T_g T_z$ has rank ≤ 1.

Proof.
$T_z T_g - T_g T_z = (PM_z PM_g - PM_z M_g)|_{H^2\mathbb{S}} = (PM_z P^\perp)M_g|_{H^2\mathbb{S}}$,
and $PM_z P^\perp$ has rank 1. □

Corollary. If $g \in L^\infty(\mathbb{S})$, $f \in C(\mathbb{S})$ then $T_f T_g - T_{fg}$ and $T_g T_f - T_{fg}$ are compact operators.

Toeplitz algebra

Set $\mathcal{T} = \{T_f + k : f \in C(\mathbb{S}),\ k \in \mathbb{K}(H^2(\mathbb{S}))\}$.

Lemma. $\mathcal{T}$ is a C^*-algebra.

Proof. $\mathcal{T}$ is closed under the operations of a $*$-algebra, e.g. $(T_{f_1} + k_1)(T_{f_2} + k_2) = T_{f_1 f_2} + k$ for some compact operator k, so it remains to check that it is norm-closed. Let $\{T_{f_n} + k_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence. Then $\|f_n - f_m\| = \|T_{f_n} - T_{f_m}\|_{ess} \le \|(T_{f_n} + k_n) - (T_{f_m} + k_m)\|$, hence the sequence $\{f_n\}$ is a Cauchy sequence as well, thus it has a limit $f = \lim_{n\to\infty} f_n \in C(\mathbb{S})$. Then $\lim_{n\to\infty} T_{f_n} = T_f$, and therefore the sequence $\{k_n\}$ is also a Cauchy sequence. As compact operators are norm-closed, so it also has a limit $k = \lim_{n\to\infty} k_n$. Thus $\lim_{n\to\infty} T_{f_n} + k_n = T_f + k \in \mathcal{T}$. □

Another way to define the Toeplitz algebra is to take the minimal unital C^*-subalgebra of $\mathbb{B}(H^2(\mathbb{S}))$ containing T_z. Then it is an excercise to check that all compact operators lie in it.

The subset $\mathbb{K} = \mathbb{K}(H^2(\mathbb{S})) \subset \mathcal{T}$ is an ideal.

Lemma. The quotient $\mathcal{T}/\mathbb{K}$ is $*$-isomorphic to $C(\mathbb{S})$.

Proof. Note that the quotient is commutative, hence it is $C(X)$ for some X to be determined.
There is a map $q : \mathcal{T} \to C(\mathbb{S})$ given by $q(T_f + k) = f$. It is obviously a $*$-homomorphism. As $\|T_f\| = \|T_f\|_{ess} = \|f\|$, so $\operatorname{Ker} q = \mathbb{K}$. It remains to check surjectivity of q. Note that there is a map $s : C(\mathbb{S}) \to \mathcal{T}$, given by $s(f) = T_f$. This map is not a $*$-homomorphism, but it is continuous, and $q \circ s = \mathrm{id}_{C(\mathbb{S})}$. □

Recall that an operator is Fredholm if it is invertible modulo the compact operators. If $f \in C(\mathbb{S})$ never equals 0 then the winding number $\operatorname{wind} f$ is well-defined as that of the curve $f(\mathbb{S})$ around 0.

T_f is Fredholm iff 0 is not in the range of f. In this case, $\operatorname{ind} T_f = -\operatorname{wind} f$.

Proof. Homotopy invariance of both *ind* and *wind* show that it suffices to check this equality for $f(z) = z^n$, where it is easy. □

Lemma. There is no $*$-homomorphism $C(\mathbb{S}) \to \mathcal{T}$ right-inverse to q.

Proof. Suppose $\sigma : C(\mathbb{S}) \to \mathcal{T}$ is a $*$-homomorphism. Commutativity implies that $\sigma(f)$ is normal for any $f \in C(\mathbb{S})$, hence $\operatorname{ind} \sigma(f) = 0$. But if $q \circ \sigma = \mathrm{id}_{C(\mathbb{S})}$ then $\operatorname{ind} \sigma(f) = -\operatorname{wind} f$. □

The ideal $\mathbb{K}$, the quotient $C(\mathbb{S})$ and the Toeplitz algebra can be organized into a short exact sequence $0 \to \mathbb{K} \to \mathcal{T} \to C(\mathbb{S}) \to 0$. Short exact sequences are often called extensions of the quotient by the ideal. The extension given by the Toeplitz algebra plays an important role in the index theory and in the K-theory of C^*-algebras.

Group C^*-algebras

Let G be a countable discrete group, and let $\pi : G \to U(H)$ be its unitary representation on a Hilbert space H. It can be extended to a $*$-representation of the group ring $\mathbb{C}[G]$ on H. Then $\|a\|_\pi = \|\pi(a)\|$, $a \in \mathbb{C}[G]$, defines a semi-norm on $\mathbb{C}[G]$, which becomes a norm after we pass to the quotient $\mathbb{C}[G]/\operatorname{Ker}\pi$. This norm is obviously a C^*-norm. The completion of $\mathbb{C}[G]/\operatorname{Ker}\pi$ with respect to $\|\cdot\|_\pi$ gives a C^*-algebra $C^*_\pi(G)$ of G determined by π.

Example. Let θ be the trivial representation, which sends all group elements to 1. Then $C^*_\theta(G) \cong \mathbb{C}$ is just scalars.

The reduced group C^*-algebra

Example. Let λ be the (left) regular representation of G on $H = l_2(G)$ given by $\lambda(g)(\sum_{h\in G} \beta_h \cdot \delta_h) = \sum_{h\in G} \beta_g \delta_{gh}$, where $\delta_h \in l_2(G)$ are delta functions (orthonormal basis), and $\beta_h \in \mathbb{C}$ are coefficients. The corresponding C^*-algebra $C^*_\lambda(G)$ is called the reduced group C^*-algebra (sometimes denoted by $C^*_r(G)$).

Note that $\|\lambda(\sum_{g\in G} \alpha_g \cdot g)\| \geq \|\lambda(\sum_{g\in G} \alpha_g \cdot g)\delta_e\|_2 = \| \sum_{g\in G} \alpha_g \cdot \delta_g\| = \sum_{g\in G} |\alpha_g|^2$, where $e \in G$ denotes the neutral element, hence $\text{Ker}\,\lambda = \{0\}$ and the reduced group C^*-algebra is a completion of the group ring without taking quotients.

The full group C^*-algebra

If $\{\pi_\gamma\}_{\gamma\in\Gamma}$ is a family of unitary representations of G (labeled by elements of a set Γ) then one can take $\pi = \oplus_{\gamma\in\Gamma}\pi_\gamma$. Then $\|\cdot\|_\pi = \sup_{\gamma\in\Gamma} \|\cdot\|_{\pi_\gamma}$. If we take all representations of G we get the universal, or full group C^*-algebra $C^*(G)$.

Example. If G is finite then there is no need to complete, so $C^*(G) = C^*_r(G) = \mathbb{C}[G]$.

Example. Let $G = \mathbb{Z}$ with a generating element $a \in \mathbb{Z}$. Let $x = \sum_{n\in\mathbb{Z}} \alpha_n a^n \in \mathbb{C}[\mathbb{Z}]$. Any unitary representation of $\mathbb{Z}$ decomposes as a direct integral of irreducible representations, all of which are known: all irreducible representations of $\mathbb{Z}$ are one-dimensional, and are determined by $\pi_t(a) = e^{2\pi it}$, $t \in [0, 1]$. Therefore, $\|x\|_u = \sup_{t\in[0,1]} \|\pi_t(x)\| = \|\sum_{n\in\mathbb{Z}} \alpha_n e^{2\pi int}\|$. Hence the map $\varphi : C^*(\mathbb{Z}) \to C(\mathbb{S})$ given by $\varphi(x)(t) = \sum_{n\in\mathbb{Z}} \alpha_n e^{2\pi int}$ is an isometry (that's why it is well defined not only on $\mathbb{C}[\mathbb{Z}]$, but also on $C^*(\mathbb{Z})$). An appropriate choice of vectors in $l_2(\mathbb{Z})$ shows that $\|x\|_\lambda = \sup_{t\in[0,1]} \|\pi_t(x)\|$.

Sometimes the universal norm $\|\cdot\|_u$ coincides with the reduced (or regular) norm $\|\cdot\|_\lambda$, but in general $\|\cdot\|_u \geq \|\cdot\|_\lambda$. The identity map on $\mathbb{C}[G]$ extends to a continuous map $C^*(G) \to C^*_r(G)$, which is always surjective, but in general this map can have a non-trivial kernel, and the algebras $C^*(G)$ and $C^*_r(G)$ may be very different. In particular, if $G = \mathbb{F}_2$ is the free group on two generators then $C^*(\mathbb{F}_2)$ has infinitely many ideals of finite codimension, while $C^*_r(\mathbb{F}_2)$ is simple.

Theorem. The surjective $*$-homomorphism $C^*(G) \to C^*_r(G)$ is a $*$-isomorphism iff G is an amenable group.

Proof: difficult. Cf. book by N. Brown and N. Ozawa.

Let A be a C^*-algebra, G a countable discrete group acting on A by automorphisms due to a homomorphism $\alpha : G \to \mathrm{Aut}(A)$. The triple (A, G, α) is called a C^*-dynamical system. A covariant representation of (A, G, α) is a pair (π, u), where $\pi : A \to \mathbb{B}(H)$ is a $*$-representation of A and $u : G \to U(H)$ is a unitary representation of G on the same Hilbert space H, such that $u_g\pi(a)u_g^* = \pi(\alpha_g(a))$ for any $a \in A$, $g \in G$.

Similarly to group rings, let $A[G]$ be the algebra of all finite sums $a = \sum_{g\in G} a_g g$, where $a_g \in A$. Multiplication on $A[G]$ is defined using the twisted convolution. If $b = \sum_{g\in G} b_g g$ then $ab = \sum_{g,h\in G} a_g g b_h h = \sum_{g,h\in G} a_g(g b_h g^{-1})gh = \sum_{g,h\in G} a_g\alpha_g(b_h)gh = \sum_{\gamma\in G}(\sum_{g\in G} a_g\alpha_g(b_{g^{-1}\gamma}))\gamma$.
Also $a^* = \sum_{g\in G}(a_g g)^* = \sum_{g\in G} \alpha_g^{-1}(a_g^*)g^{-1}$.

A covariant representation of (A, G, α) yields a $*$-representation of $A[G]$, and, conversely, a $*$-representation of $A[G]$, when restricted on A and on G (if A is unital), gives a covariant representation of (A, G, α).

Crossed product C^*-algebras

Let $\pi : A \to \mathbb{B}(H)$ be a $*$-representation of A, and let λ be the left regular representation of G. Form the Hilbert space $l_2(G; H)$ of all H-valued square-summable functions x on G with the norm $\|x\|_2^2 = \sum_{g \in G} \|x(g)\|^2$ and define a covariant representation $(\tilde{\pi}, u)$ of (A, G, α) on this Hilbert space by $(\tilde{\pi}(a)x)(g) = \pi(\alpha_g^{-1}(a))(x(g))$; $(u_g x)(h) = x(g^{-1}h)$, $g, h \in G$.

Define a norm on $A[G]$ by $\|a\| = \sup_\sigma \|\sigma(a)\|$, where the supremum is taken over all covariant representations of the C^*-dynamical system (A, G, α). This is correct, as $\|a\| \leq \sum_{g \in G} \|a_g\| < \infty$. The (full) crossed product C^*-algebra $A \rtimes_\alpha G$ is defined as the completion of $A[G]$ with respect to this norm.

Example. $\mathbb{C} \rtimes_\alpha G \cong C^*(G)$.

Example. If α is trivial then $C(X) \rtimes_\alpha G \cong C(X; C^*(G))$. In particular, if $X = \mathbb{S}$, $G = \mathbb{Z}$ then $C(\mathbb{S}) \rtimes_\alpha \mathbb{Z} \cong C(\mathbb{T}^2)$, where $\mathbb{T}^2$ is a two-dimensional torus.

Irrational rotation algebras

Let the action α of $\mathbb{Z}$ on $C(\mathbb{S})$ be defined by $\alpha_n(f)(z) = f(e^{2\pi i\theta n}z)$, $n \in \mathbb{Z}$, where $\theta \in [0, 1]$ is irrational. (This action is induced by the action by irrational rotation on the circle.) Then $A \rtimes_\alpha \mathbb{Z} = A_\theta$ is called an irrational rotation algebra, or a noncommutative torus. It can be described in terms of generating elements and relations as follows: Let $u \in C(\mathbb{S})$ denote the identity function $u(z) = z$, $z \in \mathbb{S}$, and let v denote a generator of $\mathbb{Z}$. They can be considered as unitary elements of $C(\mathbb{S}) \rtimes_\alpha \mathbb{Z}$, and they satisfy the relation $vu = e^{2\pi i\theta}uv$.

Two unitaries with this relation determine A_θ as a certain universal C^*-algebra, like two commuting unitaries determine $C(\mathbb{T}^2)$. A_θ is simple. (Difficult. Cf. the book by K. Davidson.)

By a projection in a C^*-algebra A we always mean a selfadjoint idempotent ($p = p^* = p^2$). Two projections, $p, q \in A$, are

- Murray–von Neumann equivalent ($p \sim_{MN} q$) if there exists $v \in A$ such that $p = v^*v$, $q = vv^*$;
- unitarily equivalent ($p \sim_u q$) if there exists a unitary $u \in A^+$ such that $p = u^*qu$;
- homotopy equivalent ($p \sim_h q$) if there exists a continuous path of projections that connects p and q.

There may be several other notions of equivalence, but they usually reduce to these. For example:

Lemma. If $q = zpz^{-1}$ for some $z \in A^+$ then $p \sim_u q$.

Proof. It follows from $zp = qz$ and $z^*q = pz^*$ that $pz^*z = z^*qz = z^*zp$, and as p commutes with z^*z, so it commutes with $(z^*z)^{-1/2}$. Set $u = z(z^*z)^{-1/2}$. Then u is unitary and $upu^* = z(z^*z)^{-1/2}p(z^*z)^{-1/2}z^* = zp(z^*z)^{-1}z^* = qz(z^*z)^{-1}z^* = q$.

Lemma. Let $v^*v = p$ and $vv^* = q$ be projections. Then $v = vv^*v$ and $v^* = v^*vv^*$.

Proof. $\|v - vv^*v\|^2 = \|v(1-p)\|^2 = \|(1-p)v^*v(1-p)\| = \|(1-p)p(1-p)\| = 0.$ □

Lemma. If $p \sim_{MN} q$ and $1-p \sim_{MN} 1-q$ then $p \sim_u q$.

Proof. By assumption, there exist v and w such that $v^*v = p$, $vv^* = q$, $w^*w = 1-p$, $ww^* = 1-q$. Note that $v^*w = v^*q(1-q)w = 0$, similarly, $w^*v = 0$, hence $(v+w)^*(v+w) = v^*v + w^*w = p + 1 - p = 1$. Similarly, $(v+w)(v+w)^* = 1$. Thus $u = v + w$ is unitary, and $upu^* = (v+w)v^*v(v+w)^* = vv^*vv^* = q^2 = q$. □

Let v be the right shift on l_2. Then $v^*v = 1$, $vv^* = 1 - e_1$, where e_1 is the projection onto the first coordinate. This shows that the Murray–von Neumann equivalence doesn't imply unitary equivalence.

Lemma. Let $p \in A$ be a projection, and let $V = \{q \in A : q \text{ is a projection and } \|p - q\| < 1\}$. There exists a continuous map $q \mapsto u_q$ from V to the unitary group of A^+ such that $u_p = 1$ and $u_q p u_q^* = q$.

Proof. Set $v_p = 2p - 1$, $v_q = 2q - 1$, $z_q = v_q v_p + 1$. Then $qz_q = q(2q-1)(2p-1)+q = 2qp = (2q-1)(2p-1)p+p = z_q p$. It remains to check invertibility of z_q, which follows from the estimate $\|\frac{z_q}{2} - 1\| = \frac{1}{2}\|v_q v_p + 1\| = \frac{1}{2}\|v_q(v_p - v_q)\| \leq \frac{1}{2}\|v_p - v_q\| = \|p - q\| < 1$. □

Corollary. If $\|p - q\| < 1$ then $p \sim_h q$.

Proof. As $u_p = 1$, so u_q lies in the path connected component of 1 in the unitary group, and there is a path $u(t)$ connecting 1 with u_q. Then $p(t) = u(t)pu(t)^*$ gives a required path of projections. □

Lemma. If $p \sim_h q$ then $p \sim_u q$.

Proof. Divide the homotopy path connecting p with q by intermediate points $p = p_0, p_1, p_2, \ldots, p_n = q$ such that $\|p_{i+1} - p_i\| < 1$. All projections p_i are unitarily equivalent. □

Lemma. (1) If $p \sim_{MN} q$ then $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$.
(2) If $p \sim_u q$ then $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$.

Proof. (1) If $p = v^*v$, $q = vv^*$, then set $u = \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix}$. An easy calculation shows that u is unitary in $M_2(A)$ and that $u \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} u^* = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$. □

(2) Let $q = upu^*$ with $u \in A^+$ unitary. The unitary group of a C^*-algebra need not to be path-connected, but elements of the form $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$ lie in the connected component of $1_{M_2(A^+)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The required path of unitaries is given by
$w_t = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$. Then $w_t \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} w_t^*$ is the path connecting the projections.

Idea: consider projections not in A, but in matrices of arbitrary size over A. Advantages:

- All equivalences for projections become the same;
- One can take direct sums of projections.

Set $M_\infty(A) = \cup_{n\in\mathbb{N}} M_n(A)$, and let $V(A)$ denote the set of equivalence classes of projections in $M_\infty(A)$.
Addition $[p] + [q] := \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\right]$ is well-defined and makes $V(A)$ an abelian semigroup with the trivial element $[0]$.

Example. If A is $\mathbb{C}$, M_n or $\mathbb{K}$ then the equivalence class of a projection in $M_k(A)$ is determined by its rank, hence $V(A) = \{0, 1, 2 \ldots\}$.
Example. If $A = \mathbb{B}(H)$ with H infinitedimensional then, besides projections of finite rank, there is one more equivalence class consisting of infinite rank projections, so $V(A) = \{0, 1, 2, \ldots, \infty\}$.
Example. If $A = C(X)$ then $V(A)$ can be identified with isomorphism classes of locally trivial vector bundles over X.

Stable equivalence: Two projections, $p, q \in M_\infty(A)$ are stably equivalent if there exists a projection r such that $[p+r] = [q+r]$.

Consider all pairs $([p],[q])$, $[p],[q] \in V(A)$. Two such pairs, $([p],[q])$ and $([p'],[q'])$ are equivalent if $p+q'$ and $p'+q$ are stably equivalent. The set of equivalence classes of such pairs becomes an abelian group with $([p],[q]) + ([p']+[q']) = ([p]+[p'],[q]+[q'])$, $-([p],[q]) = ([q],[p])$, and with the neutral element $([p],[p])$.

If a C^*-algebra A is unital then define $K_0(A)$ as the group of equivalence classes of pairs of projections.

If $\varphi : A \to B$ is a $*$-homomorphism of C^*-algebras then it extends to a $*$-homomorphism $\tilde{\varphi} : M_n(A) \to M_n(B)$, and $\tilde{\varphi}(p)$ is a projection in $M_n(B)$, hence K_0 is a covariant functor (even for non-unital $*$-homomorphisms). We write $\varphi_* : K_0(A) \to K_0(B)$.

$K_0(\mathbb{C}) \cong K_0(M_n) \cong \mathbb{Z}$; $K_0(\mathbb{B}(H)) = 0$.

Non-unital C^*-algebras may have no non-trivial projections at all (cf. $C_0(\mathbb{R}^2)$).

If A is not unital then there is a quotient map $\theta : A^+ \to \mathbb{C}$ with the kernel A. Set $K_0(A) = \operatorname{Ker} \theta_*$.

If B is non-unital, $\varphi : A \to B$ a $*$-homomorphism, then the composition $K_0(A) \xrightarrow{\varphi_*} K_0(B^+) \xrightarrow{\theta_*} K_0(\mathbb{C})$ is zero map, hence $\varphi_*(K_0(A)) \subset K_0(B)$, and K_0 is a functor for non-unital C^*-algebras as well.

Continuity of K_0

Lemma. Let $A_1 \subset A_2 \subset \cdots \subset A$ with $\cup_{n\in\mathbb{N}} A_n$ dense in A. Then $K_0(A) = \lim K_0(A_n)$.

Idea of proof. If p is a projection in A (or in $M_k(A^+)$) then, for any $\varepsilon > 0$ there exists some selfadjoint x in A_n (or in $M_k(A^+)$) for some k, such that $\|p - x\| < \varepsilon$. This x is not a projection, but $\|x - x^2\| < 3\varepsilon$, hence $\mathrm{Sp}\, x$ consists of two distinct pieces — one close to 0, and another one close to 1. Applying a function f which equals 0 on $(-\infty, 1/2)$ and 1 on $(1/2, \infty)$ (which is continuous on $\mathrm{Sp}\, x$), we obtain a projection $f(x)$, which is also close to p (up to 4ε).
As this procedure is continuous, it can be applied to homotopies of projections as well.