Foliated toral solenoids

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Glances at Manifods

General info

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Contents

Foliated spaces

Definition

A (compact) foliated space (lamination) X is equipped with a family $\mathcal{F}$ of pairwise disjoint connected smooth manifolds (called leaves) of a given dimension k (called dimension of $\mathcal{F}$) which fill X and locally look like parallel k-dimensional hyperplanes in $\mathbb{R}^p \times Z$, Z being a "good" (locally compact, second countable, metrizable ...) space Z (called the transverse model). If M is a manifold and $Z = \mathbb{R}^m$, then $\mathcal{F}$ is called just a foliation.

Examples

Trajectories of nowhere vanishing vector field, fibers of submersions, suspensions of homeomorphism groups (called foliated bundles), Reeb foliation, ...

Geometric entropy

For $\mathcal{F}$ equipped with a leafwise Riemannian metric:

Definition

Two points x and y of (a piece of) Z (embedded transversally to $\mathcal{F}$) are (R, ϵ)-separated whenever there exists a leaf curve γ_x of origin x and length $\leq R$ which is followed by another curve γ_y of origin y and such that $d(\gamma_x(1), \gamma_y(1)) \geq \epsilon$. If X is compact, the maximal number $N(R, \epsilon)$ of pariwise (R, ϵ)-separated points is finite and one can define the geometric entropy $h(\mathcal{F})$ of $\mathcal{F}$ by $h(\mathcal{F}) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(R, \epsilon)$.

Examples

Geometric entropy of the Reeb foliation equals zero, of the Hirsch foliation is positive, of the 1-dimensional foliation is equal to the doubled topological entropy of the time-one map of the unit tangent flow.

Distality

Definition

A group G of homeomorphisms of a metric space (X, d) is said to be distal if for any distinct points x and y of X, the distances $d(g(x), g(y))$, $g \in G$, are bounded away from zero.

Theorem

(Furstenberg, 1963) *Any minimal distal action of a group G on a compact metric space X can be expressed as (X_η, G), where η is an ordinal, (X_0, G) is the trivial action on a singleton, $(X_{\xi+1}, G)$ $(\xi < \eta)$ is an isometric extension of (X_ξ, G) and (X_ξ, G) is the limit of the family (X_ζ, G) , $\zeta < \xi$ when ξ is a limit ordinal $\leq \eta$.*

Definition

Distal foliations (or, laminations) are those with distal holonomy pseudogroups and similarly distal foliated bundles are those with distal (global) holonomy groups.

Distality and entropy

Theorem

(A. Biś, P. W., 2011) *The geometric entropy of a compact minimal distal foliated bundle vanishes if only its holonomy group has linear growth.*

Theorem

(P. W., 2013) *Any minimal distal* C^1*-foliated bundle has polynomial expansion growth and zero entropy.*

Lemma

(A. Biś, P. W., 2011) *If* (Y, G) *is a limit of the family* (Z_α, G) *and* G *has zero entropy on* Z_α *for all* α*, then* G *has zero entropy on* Y*.*

Lemma

(P. W., 2013) *If* X *and* Y *are compact metric spaces and* $\pi : X \to Y$ *a locally trivial bundle with the typical fibre* F, F *being a compact metric space of finite Hausdorff dimension* s,
(∗) *there exists a constant* $c > 0$ *such for any* $\delta > 0$ F *contains a* δ*-net of cardinality* $\leq c\delta^{-s}$,
G *is a finitely generated group acting simultaneously on* X *and* Y *in such a way that*

$$\pi(g(x)) = g(\pi(x)),\ x \in X,\ g \in G,$$

G *acts between the fibres by isometries and all the fibres are isometric to a compact metric space* F, *then the inequality*

$$N(n, \epsilon, X) \leq (m+1)\delta^{-s} \cdot N(n, \delta, Y)$$

holds with some $\epsilon \to 0$ *together with* $\delta \to 0$.

Solenoids

Definition

Given a closed manifold M, an M-solenoid is the inverse limit $\mathcal{S}$ of a sequence

$$\ldots \longrightarrow M_k \xrightarrow{\pi_k} M_{k-1} \longrightarrow \ldots M_1 \xrightarrow{\pi_1} M_0 = M,$$

$\pi_k : M_k \to M_{k-1}$ being covering maps of finite (prime) degree n_k. The sequence $\mathcal{P} = (\pi_k; k = 0, 1, 2, \ldots)$ is called presentation of $\mathcal{S}$.

Examples

(1) If $M = S^1$, then $\mathcal{S}$ is (so called) Vietoris solenoid (Vietoris, 1927). Two Vietoris solenoids $\mathcal{S}_1$ and $\mathcal{S}_2$ are homeomorphic if and only if their presentationss $\mathcal{P}_1$ and $\mathcal{P}_2$ are cofinitely equal up to a permutation. (Aarts-Foking 1991, Bing 1960,McCord 1965).
(2) Every M-solenoid can be considered as a Cantor-set foliated bundle $\mathcal{F}$ over M.

Lifting a foliation $\mathcal{F}$ of M to consecutive coverings and passing to the (inverse) limit we get a foliated M-solenoid whose leaves are solenoids (with their intrinsic laminations) transverse to another solenoid

Problem

Classify foliated T^2-solenoids.

Theorem

(Kneser, 1923) Every C^2-foliated non-minimal torus T^2 splits into the countably many zones $S^1 \times [0,1]$ of 3 types: (i) foliated by circles, (ii) foliated by spirals, (iii) Reeb components; the number of zones of type (iii) is finite.

Examples

If $T^2 = C_1 \times C_2$ (C_i's being circles), $\mathcal{S} = \mathcal{V}_1 \times \mathcal{V}_2$, $\mathcal{V}_i$ being Vietoris solenoids obtained from C_i's, then the classification is rather easy: the limit structures are equivalent iff the Vietoris solenoids are homeomorphic and the starting foliations are equivalent (under a homeomorphism preserving the product structure of T^2).

In general? In higher dimensions? in progress? Possible at all? (see, for example, papers by G. Hjorth and S. Thomas 2006, A. Clarck and S. Hurder 2011; in the last one, one can find the following:

Theorem

If S is a T^k-solenoid S and $q \geq 2k$, then there exists a distal C^0-foliation $\mathcal{F}$ of $T^k \times D^q$ such that $\mathcal{S}$ embeds as a minimal set of $\mathcal{F}$.

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Best thanks to the organizers!!!

Birthday greetings to $\forall$ of you born in

$$2015 - x,$$

where x satisfies

$$x^2 - 125x + 3900 = 0!!!$$