Remarks on the weak isovariant Borsuk-Ulam theorem

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The study of Borsuk-Ulam type theorems has long history and many generalizations of the Borsuk's antipodal theorem are obtained.

For example, the following is well known.

Theorem 1 (Borsuk-Ulam theorem).

Let G be ${C_p}^k$, p prime, or T^k. Suppose that G acts fixed-point-freely on spheres S^m, S^n. If there exists a G-map $f : S^m \to S^n$, then $m \leq n$.

In this context, Bartsch studied the groups for which a Borsuk-Ulam type result holds.

In this talk, I would first like to recall Bartsch's results, and next discuss isovariant counterparts of them.

The Borsuk-Ulam function

Let G be a compact Lie group.

Consider the unit sphere SV, called a linear G-sphere, of a (finite dimensional) orthogonal G-representation with $V^G = 0$.

Definition.

The Borsuk-Ulam function $\phi_G : \mathbb{N} \to \mathbb{N}$ is defined as follows. $\phi_G(n)$ is the maximal integer k such that the existence of a G-map from SV to SW ($V^G = W^G = 0$) with $\dim V \geq n$ implies $\dim W \geq k$.

Example.

(1) If $G = {C_2}^k$, then $\phi_G(n) = n$.
(2) If $G = {C_p}^k$ (p : odd prime) or T^k, then $\phi_G(n) = n$ if n is even, and $\phi_G(n) = n + 1$ if n is odd. (Note: in this case, $\dim V$ ($V^G = 0$) is even.)

We can see

1. ϕ_G is weakly monotone increasing.
2. $\phi_G(n+m) \leq \phi_G(n) + \phi_G(m)$ (subadditivity).
3. $\phi_G(\dim V) \leq \dim W$ if there exists a G-map $f : SV \to SW$. In particular, if we take $id : SV \to SV$, then we see that $\phi_G(n) \leq n$ for $n \in D_G$, where $D_G = \{\dim V \mid G\text{-representation } V \text{ with } V^G = 0\}$.
4. The Borsuk-Ulam theorem holds for linear G-spheres if and only if $\phi_G(n) = n$ for any $n \in D_G$.

Definition.

We say that the *weak* Borsuk-Ulam theorem holds for linear G-spheres if $\lim_{n\to\infty} \phi_G(n) = \infty$.

Theorem 2 (Bartsch).

Let G be a finite group. The following are equivalent.

(a) $\lim_{n\to\infty} \phi_G(n) = \infty$.

(b) *If there exists a G-map $f' : SV' \to SW'$ $(V'^G = W'^G = 0)$ with* $\dim V' = \infty$, *then* $\dim W' = \infty$.

(c) *If there exists a G-map $f : SV \to SW$ $(W \subset V$ and $V^G = 0)$, then $V = W$. (Waner type Borsuk-Ulam theorem).*

(d) *G is a p-group.*

A similar result is known for a compact Lie group,

Theorem 3 (Bartsch).

Let G is a compact Lie group. The following hold.

$$(a) \Longrightarrow (b) \iff (c) \iff (d').$$

(d'): G is a p-toral group ,i.e, $1 \to T \to G \to P \to 1$ (ex).

Question: Does (d') $\Longrightarrow$ (a) hold? Namely, is the weak Borsuk-Ulam theorem hold for an arbitrary p-toral group?

The Borsuk-Ulam constant

We introduce the Borsuk-Ulam constant related to the Borsuk-Ulam function.

Definition.

The Borsuk-Ulam constant b_G is defined by the supremum of $b \in \mathbb{R}$ such that if there exists a G-map $f : SV \to SW$ ($V^G = W^G = 0$), then $b \dim V \leq \dim W$. (If $G = 1$, then set $b_G = 1$ for convention.)

By definition, we have

1. $0 \leq b_G \leq 1$, and $b_G = 1$ if and only if the Borsuk-Ulam theorem holds.
2. $b_G n \leq \phi_G(n) \leq n$ for $n \in D_G$.
3. $b_G = \lim_{n\to\infty} \dfrac{\phi_G(n)}{n} \left(= \inf_n \{\dfrac{\phi_G(n)}{n}\}\right)$. (Note that the right hand side converges by subadditivity.)

Bartsch also showed the following results.

Theorem 4 (Bartsch).

1. *If* $G = C_{p^k}$, *then* $b_G = \frac{1}{p^{k-1}}$.
2. *If* G *is a non p-toral group, then* $b_G = 0$.
   (Question: Does the converse hold?)
3. *In case of finite* G, $b_G = 1$ *if and only if* $G = {C_p}^k$.

Determination of the Borsuk-Ulam function or constant is a difficult problem. However Bartsch conjectures that if G is a finite p-group, then $b_G = \frac{p}{\exp(G)}$, where $\exp(G) = \mathrm{lcm}\{o(g) \,|\, g \in G\}$. If this is true, then the above question is true when G is finite.

The Borsuk-Ulam theorem for Isovariant maps

Now we turn to the isovariant version of the Borsuk-Ulam theorem, which was first studied by Wasserman.

Definition.

A G-map $f : X \to Y$ is called G-isovariant if f preserves the isotropy subgroups: $G_x = G_{f(x)}$ ($\forall x \in X$).

In other words, it is a G-map such that $f_{|G(x)} : G(x) \to Y$ is injective on each orbit $G(x)$ of $x \in X$.

Wasserman showed the following.

Theorem 5 (Isovariant Borsuk-Ulam theorem).

Let G be solvable compact Lie group. If there exists a G-isovariant map $f : SV \to SW$, then

$$\dim V - \dim V^G \leq \dim W - \dim W^G.$$

Like the Borsuk-Ulam function, we can define the isovariant Borsuk-Ulam function.

Definition.

The isovariant Borsuk-Ulam function $\varphi_G : \mathbb{N} \to \mathbb{N}$ is defined as follows. $\varphi_G(n)$ is the maximal integer k such that the existence of a G-isovariant map from SV to SW with $\dim V - \dim V^G \geq n$ implies $\dim W - \dim W^G \geq k$.

Definition.

The isovariant Borsuk-Ulam constant c_G is defined by the supremum of $c \in \mathbb{R}$ such that if there exists a G-isovariant map $f : SV \to SW$, then $c(\dim V - \dim V^G) \le \dim W - \dim W^G$.

By definition, we have

1. $0 \le c_G \le 1$, and
   $c_G = 1$ if and only if the isovariant Borsuk-Ulam theorem holds.
   In the case, G is called a Borsuk-Ulam group (BUG for short) according to Wasserman.
2. $c_G n \le \varphi_G(n) \le n$ for $n \in D_G$.
3. $c_G = \lim_{n\to\infty} \frac{\varphi_G(n)}{n} \; (= \inf_n \{\frac{\varphi_G(n)}{n}\})$.

Definition.

We say that the weak isovariant Borsuk-Ulam theorem holds for linear G-spheres if $\lim_{n\to\infty} \varphi_G(n) = \infty$.

Theorem 6 (N).

For an arbitrary compact Lie group G, the following hold.

1. $c_G > 0$, *and so* $\lim_{n\to\infty} \varphi_G(n) = \infty$.
2. *If there exists a G-isovariant map $f' : SV' \to SW'$ with* $\dim(V' - V'^G) = \infty$, *then* $\dim(W' - W'^G) = \infty$.
3. *If there exists a G-isovariant map $f : SV \to SW$ ($W \subset V$ and $V^G = 0$), then $V = W$.*

Thus the weak isovariant Borsuk-Ulam theorem holds for any G.

Examples of $c_G = 1$

There are many examples of finite Borsuk-Ulam groups. We recall Wasserman's and our results.

Definition (Prime condition (PC)).

1. We say that a finite simple group G satisfies (PC) if

$$\sum_{p \mid o(g)} \frac{1}{p} \leq 1 \ \text{ for } \ \forall g \in G,$$

   where $o(g)$ is the order of g, and the sum is taken over all prime divisors of $o(g)$.

2. We say that a finite group G satisfies (PC) if, for a composition series

$$1 = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_r = G,$$

   each simple H_i/H_{i-1} satisfies (PC) in the sense of (1).

Theorem 7 (Wasserman).

If a finite group G satisfies (PC), then $c_G = 1$, i.e., G is a BUG.

Clearly $G = C_p$ satisfies (PC) and so a solvable group satisfies (PC). Thus, as mentioned before, we obtain the isovariant Borsuk-Ulam theorem, namely,

Corollary 8.

For a finite solvable group G, $c_G = 1$.

Also (PC) gives several nonsolvable examples.

Example.

1. If $n < 12$, then the alternating group A_n satisfies (PC), and hence a BUG.
2. If p is a prime < 59, then $\mathrm{PSL}(2, p)$ satisfies (PC), and hence a BUG.

and so on.

Remark.

1. If $n \geq 12$, then A_n does not satisfy (PC). (Question: Is A_n a BUG for $n \geq 12$?)
2. $\mathrm{PSL}(2, 59)$, $\mathrm{PSL}(2, 61)$ etc. do not satisfy (PC), but these are BUGs by a result of N. and Ushitaki below.

Some families of BUGs

We have found the following families of BUGs by using finite group theory and character theory. These include some BUGs not satisfying (PC) as above.

Let G_p denote a Sylow p-subgroup of G.

Theorem 9 (N-Ushitaki).

If G satisfies one of the following conditions, then $c_G = 1$.

1. $G_2 = C_{2^r}$: *cyclic. (In this case, G is solvable.)*
2. $G_2 = D_{2^r}$: *dihedral group of oder 2^r $(r \geq 2)$.*
   As a convention, $D_4 = C_2 \times C_2$,
   e.g. $\mathrm{PSL}(2, q)$, q: a power of odd prime.

Theorem 9 (N-Ushitaki).

3. *$G_2 = Q_{2^r}$: generalized quaternion group of order 2^r ($r \geq 3$), e.g. $\mathrm{SL}(2, q)$, q: a power of odd prime.*
4. *G_2 is abelian and G_p is cyclic for p: odd, e.g. $\mathrm{SL}(2, 2^r) = \mathrm{PSL}(2, 2^r)$.*

Remark.

At present, a non Borsuk-Ulam group is not known. Wasserman conjectures that $c_G = 1$ for all finite groups.

The case of compact Lie groups

Next we consider the case of (connected) compact Lie groups.

Proposition 10.

An n-torus T^n is a BUG, i.e. $c_{T^n} = 1$.

This easily follows from the Borsuk-Ulam theorem for T^n.

However, other connected BUGs are not known (as far as I know).

Here we would like to try to estimate c_G for simple compact Lie groups G of types $A_n - D_n$ after preparing some properties of c_G.

(For exceptional compact Lie groups, similar results are expected.)

Basic properties of c_G

Let G be an arbitrary compact Lie group G. By representation theory, the following properties hold.

Proposition 11.

1. *If* $1 \to K \to G \to Q \to 1$ *is exact, then*
$$\min\{c_K, c_Q\} \le c_G \le c_Q.$$
In particular, if $c_K = 1$*, then* $c_G = c_Q$*.*
2. *If* $1 = K_0 \triangleleft K_1 \triangleleft K_2 \triangleleft \cdots \triangleleft K_r = G$*, then*
$$\min_{1\le i\le r}\{c_{K_i/K_{i-1}}\} \le c_G.$$

Also we have

Proposition 12.

If $G = G_1 \times G_2$*, then* $c_G = \min\{c_{G_1}, c_{G_2}\}$*, and inductively we have*

$$c_{G_1 \times \cdots \times G_r} = \min_i \{c_{G_i}\}.$$

Proof.

Consider two exact sequences: $1 \to G_1 \to G \to G_2 \to 1$ and $1 \to G_2 \to G \to G_1 \to 1$, and apply Proposition 11.

When G is *connected*, by the structure theorem, G has a form

$$G = (T \times G_1 \times \cdots \times G_r)/K,$$

where each G_i is a simple compact Lie group and K is a finite subgroup of the center of $T \times G_1 \times \cdots \times G_r$.

Using the above properties, we have

Proposition 13.

If G is connected, then $c_G = \min_i\{c_{G_i}\}$, where G_i are simple components of G.

Example.

$$c_{\mathrm{U}(n)} = c_{\mathrm{SU}(n)}.$$

In fact $\mathrm{U}(n) \cong (S^1 \times \mathrm{SU}(n))/K$, where $K \cong C_n$.

Estimation of c_G

In order to estimate c_G, we introduce a constant d_G for a connected compact Lie group G. Let T be a maximal torus of G.

Definition.

$$d_G = \sup\left\{ \frac{\dim U^T}{\dim U} \;\middle|\; U : \text{nontrivial irreducible } G\text{-representation} \right\}.$$

Clearly $0 \leq d_G \leq 1$ and it is easily seen that

$$\dim V^T \leq d_G \dim V$$

for any G-representation V with $V^G = 0$.

Proposition 14.

$$c_G \geq 1 - d_G.$$

Proof.

Let $f : SV \to SW$ be a G-isovariant map between G-representation spheres and $\tilde{f} : V \to W$ the radial extension of f. Decompose

$$V = V_\perp \oplus V^G, \qquad W = W_\perp \oplus W^G,$$

where $V_\perp$ [resp. $W_\perp$] denotes the orthogonal complement of V^G [resp. W^G] in V [resp. W].
Then the composition

$$g : V_\perp \xrightarrow{i} V \xrightarrow{\tilde{f}} W \xrightarrow{p} W_\perp$$

is a G-isovariant map and induces a G-isovariant map $\bar{g} : SV_\perp \to SW_\perp$ by normalization.

Proof (continued).

Since $\mathrm{Res}_T\, \bar{g} : SV_\perp \to SW_\perp$ is a T-isovariant map and T is a Borsuk-Ulam group (i.e., $c_T = 1$), we have

$$\dim V_\perp - \dim {V_\perp}^T \leq \dim W_\perp - \dim {W_\perp}^T.$$

On the other hand, $\dim {V_\perp}^T \leq d_G \dim V_\perp$. Hence

$$(1 - d_G) \dim V_\perp \leq \dim W_\perp - \dim {W_\perp}^T \leq \dim W_\perp.$$

This implies

$$(1 - d_G)(\dim V - \dim V^G) \leq \dim W - \dim W^G,$$

and so $c_G \geq 1 - d_G$. □

Estimates of c_G for linear compact Lie groups G.

Using representation theory, in particular, Freudenthal's multiplicity formula, we can estimate d_G. Thus we have

Theorem 15.

1. *If G is of type A_n $(n \geq 1)$, e.g. $\mathrm{SU}(n+1)$, then* $c_G \geq \dfrac{n+1}{n+2}$.
2. *If G is of type B_n $(n \geq 2)$, e.g. $\mathrm{SO}(2n+1)$, then* $c_G \geq \dfrac{n}{n+1}$.
3. *If G is of type C_n $(n \geq 3)$, e.g. $\mathrm{Sp}(n)$, then* $c_G \geq \dfrac{2n+1}{2n+3}$.
4. *If G is of type D_n $(n \geq 4)$, e.g. $\mathrm{SO}(2n)$, then* $c_G \geq \dfrac{2n-1}{2n+1}$.

Note. (1) It is observed that $c_G \to 1$ as $n \to \infty$ in each case.

(2) These estimates are not best possible. For example, a further argument shows that $c_{\mathrm{U}(2)} = c_{\mathrm{SU}(2)} \geq \frac{4}{5}$.