An algebraic description of the symbol of a linear differential operator

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Plan

- Basic definitions and facts about vector bundles and sections
- Γ functor, Serre-Swan theorem and 1-forms
- A Linear differential operator and its symbol
- Examples how to compute the symbol

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Vector bundles and sections

Definition

Let M be a smooth manifold. A smooth manifold E with a smooth surjective map $\pi : E \to M$ is called **(smooth) vector bundle of rank k,** if

- for every $x \in M$ a set $E_x := \pi^{-1}(\{x\})$ (called **fiber over** x) has a structure of vector space of dimension k
- every $x \in M$ has a neighborhood U and diffeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ (called **local trivialization**) such that $\pi_1 \circ \phi = \pi$ (here π_1 denotes projection on the first component $\pi_1 : U \times \mathbb{R}^k \to U$)
- restriction of ϕ to each fiber, $\phi : E_x \to \{x\} \times \mathbb{R}^k$, is a linear isomorphism.

Definition

Let $\pi : E \to M$ be a vector bundle over M. A smooth map $s : M \to E$ is called a **(smooth) section of E**, if $\pi \circ s = \mathrm{id}_M$.

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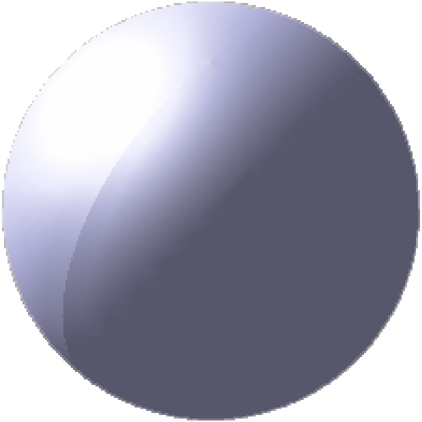

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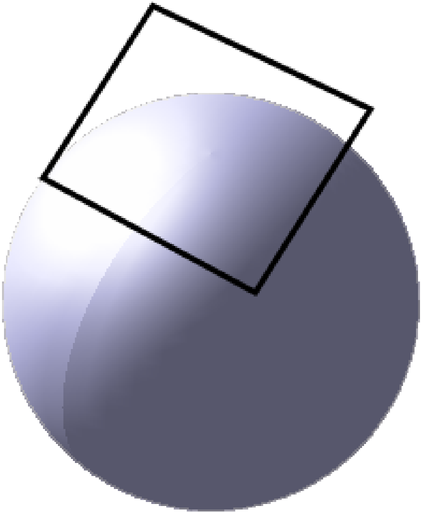

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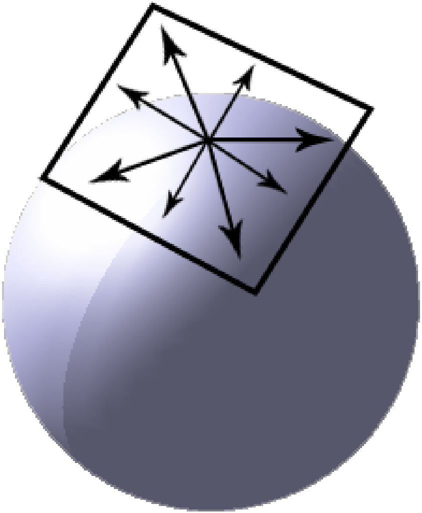

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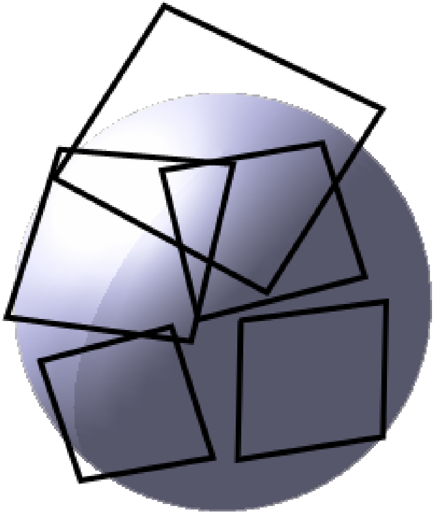

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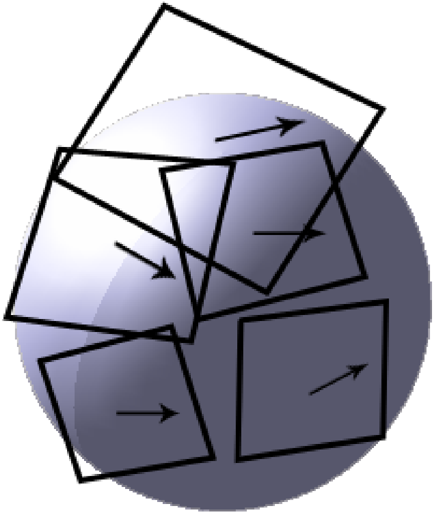

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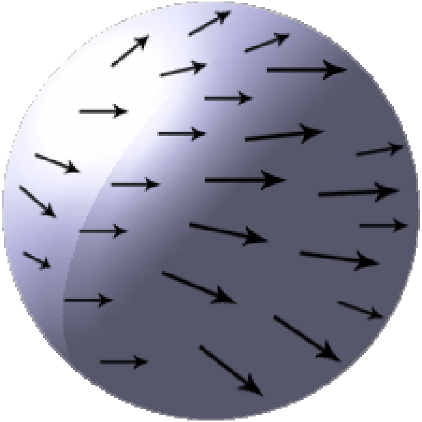

Module of sections

Theorem

Let $\pi : E \to M$ be a vector bundle over M. For any smooth sections s_1 and s_2 of E a map $s_1 + s_2 : M \to E$ defined by formula

$$(s_1 + s_2)(x) = s_1(x) + s_2(x)$$

is a smooth section.
Similarly for any smooth section s and any $f \in C^\infty(M)$ a map $fs : M \to E$ defined by formula

$$(fs)(x) = f(x)s(x)$$

is a smooth section.
Additionally the set of smooth sections with these operations form a module structure over a ring $C^\infty(M)$.

Definition

Let $\pi : E \to M$ be a vector bundle. A set of smooth sections of E with $C^\infty(M)$-module structure we call **module of sections of** E and denote $\Gamma(E)$.

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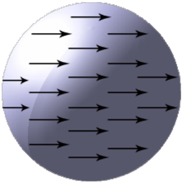
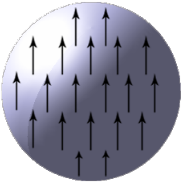
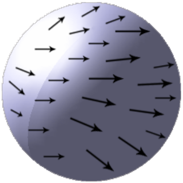
f  $+\, g$  $=$ 

Tangent bundle

Definition

Let M be a smooth manifold. Fix $x \in M$. A $\mathbb{R}$-linear map $D : C^\infty(M) \to \mathbb{R}$ is called a **tangent vector at** x, if it satisfies Leibniz rule at x, i.e. for any $f, g \in C^\infty(M)$

$$D(fg) = f(x)D(g) + g(x)D(f).$$

Vector space of tangent vectors at x is called **tangent space at** x and denoted T_xM.

Lemma

Any smooth function f in a starlike neighborhood (in $\mathbb{R}^n$) of a point z is of the form

$$f(x) = f(z) + \sum_{i=1}^{n} (x_i - z_i) \int_0^1 \frac{\partial f}{\partial x_i}(z + t(x - z))dt.$$

Theorem

If M is a manifold of dimension $n, x \in M$ and $(U, \phi) = (U, x^1, \ldots, x^n)$ is a chart containing x, then

$$\left.\frac{\partial}{\partial x^1}\right|_x, \ldots, \left.\frac{\partial}{\partial x^n}\right|_x$$

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form a basis in $T_x M$.

Tangent bundle

Theorem

Let M be a manifold of dimension n with an atlas $\{(U_\alpha, \phi_\alpha)\}_\alpha$. For each α define

$$TU_\alpha := \bigsqcup_{x \in U_\alpha} T_x M$$

and $\bar{\phi}_\alpha : TU_\alpha \to \phi_\alpha(U_\alpha) \times \mathbb{R}^n$ by formula

$$\bar{\phi}_\alpha(x, D) = (\phi_\alpha(x), a_1(x), \ldots, a_n(x)),$$

where $a_1(x), \ldots, a_n(x)$ are such that, if $\phi_\alpha = (x^1, \ldots, x^n)$, then

$$D = \sum_{i=1}^{n} a_i(x) \left. \frac{\partial}{\partial x^i} \right|_x .$$

The family $\{(TU_\alpha, \bar{\phi}_\alpha)\}_\alpha$ forms a smooth atlas on

$$TM := \bigsqcup_{x \in M} T_x M.$$

Tangent bundle

Theorem

As a consequence

$$TM = \bigsqcup_{x \in M} T_x M$$

is a smooth manifold of dimension $2n$.
Canonical projection $\pi : TM \to M$ is smooth and form a vector bundle.

Definition

If M is a smooth manifold, then TM is called **tangent bundle of M**.
Smooth sections of TM are called **(smooth) vector fields on M** and module of vector fields is denoted $\mathfrak{X}(M)$.

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Characterization of vector fields

Definition

We say that $\mathbb{R}$-linear map $X : C^\infty(M) \to C^\infty(M)$ is a **derivation**, if it satisfies Leibniz rule, i.e. for any $f, g \in C^\infty(M)$

$$X(fg) = fX(g) + gX(f).$$

The $C^\infty(M)$-module of derivations is denoted by $\mathrm{Der}(C^\infty(M))$.

Theorem

Every vector field $X \in \mathfrak{X}(M)$ gives rise to a derivation $\hat{X} \in \mathrm{Der}(C^\infty(M))$ defined by formula

$$[\hat{X}(f)](p) := X_p(f).$$

This mapping is an isomporphism between $C^\infty(M)$-modules

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Functor Γ

Definition

Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be vector bundles. A smooth map $\Phi : E \to F$ is called **(smooth) vector bundle map over** M, if $\pi_E = \pi_F \circ \Phi$ and for every $x \in M$ map $\Phi|_{E_x} : E_x \to F_x$ is $\mathbb{R}$-linear.

Theorem

Every vector bundle map $\Phi : E \to F$ over M induces $C^\infty(M)$-linear map $\Gamma(\Phi) : \Gamma(E) \to \Gamma(F)$ given by formula

$$\Gamma(\Phi)(s) = \Phi \circ s.$$

Moreover for every vector bundle maps $\Phi : E_1 \to E_2, \Psi : E_2 \to E_3$ we have that

$$\Gamma(\Psi \circ \Phi) = \Gamma(\Psi) \circ \Gamma(\Phi)$$

and for every vector bundle $\pi : E \to M$

$$\Gamma(\mathrm{id}_E) = \mathrm{id}_{\Gamma(E)}.$$

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Lemma

Let $\Phi, \Psi : E \to F$ be vector bundle maps over M. If $\Gamma(\Phi) = \Gamma(\Psi)$, then $\Phi = \Psi$.

Lemma

If $\pi : E \to M$ is a vector bundle over M, then for every $x \in M$ the sequence

$$0 \to \mu_x \Gamma(E) \hookrightarrow \Gamma(E) \xrightarrow{\pi_x} E_x \to 0$$

is exact, where $\mu_x = \{f \in C^\infty(M) : f(x) = 0\}$. Hence $\Gamma(E)/\mu_x\Gamma(E) \cong E_x$.

Theorem

Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be vector bundles. Given any $C^\infty(M)$-linear map $F : \Gamma(E) \to \Gamma(F)$ there is unique vector bundle map $\Phi : E \to F$ such that

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Theorem

Let M be a smooth manifold. A $C^\infty(M)$-module P is isomorphic to a module $\Gamma(E)$ for some vector bundle $\pi : E \to M$, if and only if module P is finitely generated and projective, i.e.

$$P \oplus Q = \bigoplus_{i=1}^{k} C^\infty(M)$$

for some $C^\infty(M)$-module Q and $k \in \mathbb{N}$.

Definition

Let M be smooth manifold. A category of smooth vector bundles over M and a category of projective finitely generated $C^\infty(M)$-modules are denoted respectively

$$\mathrm{SmoothVecBund}(M) \quad \text{and} \quad C^\infty(M)\text{-}\mathrm{FinProjMod}\,.$$

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Theorem

For every vector bundles $\pi_E : E \to M$ and $\pi_F : F \to M$

$$\Gamma(E \oplus_{\mathbb{R}} F) \cong \Gamma(E) \oplus_{C^\infty(M)} \Gamma(F),$$
$$\Gamma(E \otimes_{\mathbb{R}} F) \cong \Gamma(E) \otimes_{C^\infty(M)} \Gamma(F),$$
$$\Gamma(E \wedge_{\mathbb{R}} F) \cong \Gamma(E) \wedge_{C^\infty(M)} \Gamma(F),$$
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$$\Gamma(\mathrm{Hom}_{\mathbb{R}}(E, F)) \cong \mathrm{Hom}_{C^\infty(M)}(\Gamma(E), \Gamma(F)).$$

Corollary

For every vector bundle $\pi_E : E \to M$

$$\Gamma(E^*) \cong \Gamma(E)^\vee$$

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Cotangent bundle

Definition

Let M be a smooth manifold. Denote by T^*M the dual vector bundle to the tangent bundle TM (i.e $T^*M := TM^*$) and call it **cotangent bundle**. Sections of T^*M (elements of $\Gamma(T^*M)$) are called **1-forms**. Module of 1-forms is denoted by $\Omega^1(M)$.

Corollary

For every smooth manifold M

$$\Omega^1(M) \cong \mathfrak{X}(M)^\vee \cong \mathrm{Der}(C^\infty(M))^\vee.$$

Definition

Let M be a smooth manifold. We define a $\mathbb{R}$-linear map $d : C^\infty(M) \to \Omega^1(M)$ by formula

$$(df)(X) := X(f).$$

df is called **differential of** f.

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For any $f, g \in C^\infty(M)$

$$d(fg) = fd(g) + gd(f).$$

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$C^\infty(M)$-module of such derivations is denoted by $\mathrm{Der}(P)$.

Remark

$$d \in \mathrm{Der}(\Omega^1(M))$$

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Universal derivation

Theorem

For every finitely generated projective $C^\infty(M)$-module P and every derivation $D \in \mathrm{Der}(P)$ there exists an unique $C^\infty(M)$-linear map $\gamma : \Omega^1(M) \to P$ such that $\gamma \circ d = D$, i.e. the following diagram commutes:

$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{D} & P \\ \Big\downarrow{\scriptstyle d} & \nearrow{\scriptstyle \gamma} & \\ \Omega^1(M) & & \end{array} .$$

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Don't forget it completely!

Linear differential operators

Definition

Let M be a smooth manifold and let $\pi_E : E \to M, \pi_F : F \to M$ be smooth vector bundles over M of dimensions respectively r and s. A $\mathbb{R}$-linear map $D : \Gamma(E) \to \Gamma(F)$ is called **linear differential operator**, if there exists $k \geqslant 0$ such that for every point $x \in M$ there is a neighborhood U, that operator D locally trivialized to $L : C^\infty(U; \mathbb{R}^r) \to C^\infty(U; \mathbb{R}^s)$ is of the following form

$$L = \sum_{|\alpha| \leqslant k} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

where A^α is $r \times s$ matrix with values in smooth functions on U.
The **order of** D is the smallest k such that definition holds.

Linear differential operators

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Given two $C^\infty(M)$-modules P, R the vector space $\mathrm{Hom}_\mathbb{R}(P, R)$ has a $C^\infty(M) \otimes C^\infty(M)$-module structure defined in a way, that if $f, g \in C^\infty(M)$ and $D \in \mathrm{Hom}_\mathbb{R}(P, R)$, then

$$[(f \otimes g)D](p) := fD(gp).$$

Moreover for each $f \in C^\infty(M)$ we define $\delta(f) \in C^\infty(M) \otimes C^\infty(M)$ by:

$$\delta(f) = 1 \otimes f - f \otimes 1.$$

Remark

$D \in \mathrm{Hom}_\mathbb{R}(P, R)$ is $C^\infty(M)$-linear, if and only if for every $f \in C^\infty(M)$

$$\delta(f)D = 0.$$

In fact

$$[\delta(f)D](p) = 0 \iff D(fp) - fD(p) = 0 \iff D(fp) = fD(p).$$

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Remark

$D \in \mathrm{Hom}_{\mathbb{R}}(C^\infty(M), P)$ is a derivation, if and only if $D(1) = 0$ and for every $f, g \in C^\infty(M)$

$$[\delta(f)\delta(g)D](1) = 0.$$

In fact

$$[\delta(f)\delta(g)D](1) = 0 \iff D(fg) - fD(g) - gD(f) + fgD(1) = 0.$$

Linear differential operators

Lemma

Any smooth function f in a starlike neighborhood (in $\mathbb{R}^n$) of a point z is of the form

$$f(x) = \sum_{|\alpha| \leqslant k} \frac{1}{|\alpha|!}(x-z)^\alpha \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(z) + \sum_{|\beta|=k+1} (x-z)^\beta g_\beta(x),$$

where g_β are smooth.

Theorem

Let M be a smooth manifold and let E, F be smooth vector bundle over M. A $\mathbb{R}$-linear map $D : \Gamma(E) \to \Gamma(F)$ is a linear differential operator of order k, if and only if for every $f_0, \ldots, f_k \in C^\infty(M)$

$$\delta(f_0) \ldots \delta(f_k) D = 0$$

and there exists $f_1, \ldots, f_k \in C^\infty(M)$ such that

$$\delta(f_1) \ldots \delta(f_k) D \neq 0.$$

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Symbol

Definition

Let M be a smooth manifold, E, F be smooth vector bundles over M and let $D : \Gamma(E) \to \Gamma(F)$ be a linear differential operator of order k.
The **symbol of linear differential operator** D is a vector bundle map

$$\sigma_D : \bigodot^k T^*M \otimes E \to F$$

such that, if $x \in M$, D locally trivialized in a neighborhood U takes a form

$$\sum_{|\alpha| \leqslant k} A^\alpha \frac{\partial^\alpha}{\partial x^\alpha}$$

and if we fix $\omega_1, \ldots, \omega_k \in \Omega^1(M)$ such that locally $\omega_i = \sum_j \omega_{ij} dx^j$, then locally trivialized

$$\Gamma(\sigma_D)((\omega_1 \odot \cdots \odot \omega_k) \otimes \cdot) : \Gamma(E) \to \Gamma(F)$$

takes a form

$$\sum_{|\alpha| = k} A^\alpha \omega_\alpha .$$

Symbol

Lemma

For every $f, g \in C^\infty(M)$

$$\delta(fg) = f\delta(g) + g\delta(f) + \delta(f)\delta(g).$$

Theorem

Let M be a smooth manifold, P, R be finitely generated projective $C^\infty(M)$-modules and let $D : P \to R$ be a linear differential operator of order k. Fix $f_1, \ldots, f_{s-1}, f_{s+1}, \ldots, f_k \in C^\infty(M)$ and define a $\mathbb{R}$-linear map $\phi_{D,s} : C^\infty(M) \to \mathrm{Hom}_{C^\infty(M)}(P, R)$ by formula

$$\phi_{D,s}(f) := \delta(f_1) \ldots \delta(f_{s-1})\delta(f)\delta(f_{s+1}) \ldots \delta(f_k)D.$$

$\phi_{D,s}$ is a derivation, i.e. $\phi_{D,s}(fg) = f\phi_{D,s}(g) + g\phi_{D,s}(f)$.

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Corollary

There is unique $C^\infty(M)$-linear map $\gamma_{D,s} : \Omega^1(M) \to \mathrm{Hom}_{C^\infty(M)}(P,R)$ such that

$$\gamma_{D,s}(df) = \phi_{D,s}(f) = \delta(f_1)\dots\delta(f_{s-1})\delta(f)\delta(f_{s+1})\dots\delta(f_k)D$$

for every $f \in C^\infty(M)$.

Lemma

For every $f, g \in C^\infty(M)$

$$\delta(f)\delta(g) = \delta(g)\delta(f).$$

Corollary

There is unique $C^\infty(M)$-linear map $\gamma_D : \bigodot^k \Omega^1(M) \otimes P \to R$ such that

$$\gamma_D((df_1 \odot \cdots \odot df_k) \otimes p) = [\delta(f_1)\dots\delta(f_k)D](p)$$

for every $f_1, \dots, f_k \in C^\infty(M)$ and $p \in P$.

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Theorem

If M is a smooth manifold, E, F are smooth vector bundles and $D : \Gamma(E) \to \Gamma(F)$ is linear differential operator of order k, then

$$\Gamma(\sigma_D) \cong \gamma_D.$$

Symbol

Lemma

For every $C^\infty(M)$-modules P, Q, R, maps $D \in \mathrm{Hom}_{\mathbb{R}}(P, Q), E \in \mathrm{Hom}_{\mathbb{R}}(Q, R)$ and $f \in C^\infty(M)$

$$\delta(f)(E \circ D) = \delta(f)E \circ D + E \circ \delta(f)D.$$

Theorem

For every P, Q, R finitely generated projective $C^\infty(M)$-modules, $D \in \mathrm{Hom}_{\mathbb{R}}(P, Q)$ and $E \in \mathrm{Hom}_{\mathbb{R}}(Q, R)$ linear differential operators of orders k and l respectively the map $E \circ D$ is linear differential operator of order $k + l$, and if we define a $C^\infty(M)$-linear map

$$\mathrm{comp}_{E,D} : \bigodot^l \Omega^1(M) \otimes \bigodot^k \Omega^1(M) \to \mathrm{Hom}_{C^\infty(M)}(P, R)$$

by formula

$$\mathrm{comp}_{E,D}((\beta_1 \odot \cdots \odot \beta_l) \otimes (\alpha_1 \odot \cdots \odot \alpha_k)) = \gamma_E(\beta_1 \odot \cdots \odot \beta_l) \circ \gamma_D(\alpha_1 \odot \cdots \odot \alpha_k),$$

then

$$\gamma_{E \circ D} = \mathrm{Sym}(\mathrm{comp}_{E,D})$$

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then

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Example 1

Exterior derivative

Let $d : \Omega^*(M) \to \Omega^{*+1}(M)$ be an exterior derivative. Since for $\omega \in \Omega^s(M)$ and $\eta \in \Omega^*(M)$

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^s \omega \wedge d\eta,$$

we have that

$$(\delta(f)d)(\eta) = d(f\eta) - f d\eta = df \wedge \eta + f d\eta - f d\eta = df \wedge \eta.$$

Hence

$$\gamma_d(\alpha \otimes \eta) = \alpha \wedge \eta$$

Example 2

Definition

Let M be a n dimensional orientable Riemannian manifold with Riemannina metric $g \in \Omega^1(M) \odot \Omega^1(M)$ and volume form $\omega \in \Omega^n(M)$. Consider musical isomorphism $^\flat : \mathfrak{X}(M) \to \Omega^1(M)$ and $^\sharp : \Omega^1(M) \to \mathfrak{X}(M)$ defined be formula

$$X^\flat(Y) = g(X, Y)$$

and relation that for all $X \in \mathfrak{X}(M)$

$$\alpha(X) = g(X, \alpha^\sharp)$$

Let $* : \Omega^*(M) \to \Omega^{n-*}(M)$ be a $C^\infty(M)$-linear map defined by formula

$$*\eta = \omega \llcorner \eta^\sharp,$$

where $\llcorner$ is a tensor contraction.

Example 2

Definition

Define **Laplace operator** $\triangle : C^\infty(M) \to C^\infty(M)$ by formula

$$\triangle = *d*d.$$

$$\gamma_\triangle(\beta \odot \alpha) = * \circ \gamma_d(\beta) \circ * \circ \gamma_d(\alpha) + * \circ \gamma_d(\alpha) \circ * \circ \gamma_d(\beta).$$

Hence

$$\gamma_\triangle((\beta \odot \alpha) \otimes f) = *(\beta \wedge *(\alpha \wedge f)) + *(\alpha \wedge *(\beta \wedge f)) = f[*(\beta \wedge *\alpha) + *(\alpha \wedge *\beta)].$$

Lemma

For every $\alpha, \beta \in \Omega^1(M)$

$$g(\alpha^\sharp, \beta^\sharp) = *(\alpha \wedge *\beta).$$

Corollary

$$\gamma_\triangle((\alpha \odot \beta) \otimes f) = f[g(\beta, \alpha) + g(\alpha, \beta)] = 2f \cdot g(\alpha, \beta).$$

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