Bounded geometry and leaves

Looking for obstructions

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- if we require the transition maps to be C^∞ in the direction of the leaves then we have a C^∞-lamination. From now on, all laminations we consider are C^∞;
- in general the factors Z_i will not be manifolds, so we only have a C^0 transverse structure.

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- Inverse limits of covering maps (solenoids).

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- a Riemannian foliated space induces Riemannian metrics on every one of its leaves.

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- given a local transversal $Z \subset X$ containing x and a leafwise loop σ with basepoint x, sliding Z along σ defines the germ at x of a homeomorphism of Z;
- if all these germs are the identity, we say that X has *trivial dynamics* or *no holonomy*.

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This questions can also be posed in the context of some special kind of foliated spaces, e.g. foliations, codimension 0, etc.

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Remember: bounded geometry means positive injectivity radius and bounded covariant derivatives of the curvature.
Examples of bounded geometry: Compact manifolds, symmetric spaces, Lie groups with left Riemannian structure.
Examples of unbounded geometry: plane with arbitrarily small handles.

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- Some months ago, Robert Schmidt investigated the relation between realizability and coarse homology.

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As every C^∞-manifold admits metrics of bounded geometry (Greene), this gives also a positive answer in the general case for differential realizability.

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- Then the closure of that copy should be a compact lamination, so we can realize metrically every Riemannian manifold of bounded geometry.

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- So $\mathcal{M}_*^\infty(n)$ is both a subset and the differential analogue of the Gromov Space $\mathcal{M}_*$ of pointed proper metric spaces with Gromov-Hausdorff convergence .
- It should be noted however that C^∞ convergence and Gromov-Hausdorff convergence of manifolds only coincide on subspaces of uniformly bounded geometry (Cheeger, Lesa).

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- So we cannot expect $\mathcal{M}_*^\infty(n)$ to contain an isometric copy of every manifold of bounded geometry, since the manifolds with isometries are quotiented.

Properties of $\mathcal{M}_*^\infty(n)$-2

- If M is locally non periodic, i.e. for every point $x \in M$ there is a neighbourhood U such that the only isometry $\phi\colon M \to M$ mapping x into U is the indentity, then $\iota_M(M)$ is a topological manifold.

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- So manifolds of bounded geometry without symmetries such that its limiting manifolds are LNP can be realized as leaves of the compact lamination $\overline{\iota_M(M)}$. This is too restrictive.

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- Given a pair (M, f) there is a canonical map $\iota_{M,f}\colon M \to \widehat{\mathcal{M}}_*^\infty(n)$ sending x to (M, x, f).
- If a pair (M, f) has no isometries other than the identity, then the map $\iota_{M,f}$ is injective. For example, this is satisfied if f itself is injective.

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- $\widehat{\mathcal{M}}^{\infty}_{*,imm}(n)$ has canonical C^{∞} and Riemannian structure such that the metric induced over the subspace $\iota_{M,f}(M)$ is precisely that of M.
- If the norm of the first derivative is bounded away from zero, then $\overline{\iota_{M,f}(M)}$ is contained in $\widehat{\mathcal{M}}^{\infty}_{*,imm}(n)$.

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Example: embedding the real line into a spiral.

Further advances

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Question

Does every graph G of bounded geometry admit a finite decoration α such that any decorated graph (G', α') in the closure of (G, α) has no non-trivial decoration-preserving automorphism?

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- The space $\mathcal{M}_*^\infty(n)$ is nice (separable and completely metrizable), so we can try to use techniques of descriptive set theory.
- For example, if we can prove that some equivalence relation is a Borel subset of $\mathcal{M}_*^\infty(n) \times \mathcal{M}_*^\infty(n)$, then we can use a lot of known results.

Thank you