Metric aspects of contact manifolds

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Plan of the talk

Metric aspects: I consider contact manifolds (M, η) with fixed contact form η, and an extra assumption of the existence of Riemannian metrics “compatible” with η.
Plan:

1. Explanation of this compatibility
2. Examples of “metric contact manifolds”, that is, K-contact and Sasakian structures
3. Basic structure theorem for K-contact and Sasakian
4. Problems: relations between metric structures and topological properties
5. A survey of some recent results

Symplectic manifolds

(M^{2n}, ω), $\omega \in \Omega^2(M)$ such that $d\omega = 0$ and $\omega^n = \omega \wedge \cdots \wedge \omega \neq 0$.
An almost complex structure $J : TM \to TM$, $J^2 = -\mathrm{Id}$ is called *compatible* with ω, if

$$\omega(JX, JY) = \omega(X, Y), \text{ and } g(X, Y) = \omega(JX, Y)$$

is a Riemannian metric on M.

Theorem

For any symplectic ω there exists a compatible almost complex structure J (and a compatible Riemannian metric).

Triality:

$$(g, \omega), (J, \omega), (J, g).$$

The pair (g, J) determines an *almost symplectic* structure.

Kaehler manifolds

Assume that a pair (g, ω) determines an almost complex structure J. We say that J is integrable, if

$$[JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] = 0$$

("the Nijenhuis tensor vanishes").
If J is integrable, we say that (M, ω, J, g) is **Kaehler**.

Contact forms

Consider (closed) M^{2n+1}.

Contact forms

A contact form: $\eta \in \Omega^1(M)$ such that

$$\eta \wedge (d\eta)^n \neq 0$$

.

Associates:

- $\mathcal{F} \subset TM$ a distribution $\mathcal{F} = \mathrm{Ker}\eta$ of codimension 1;
- *Reeb vector field* $\xi : M \to TM$ on M with the properties

$$\eta(\xi) = 1, i_\xi d\eta = 0.$$

Existence of the Reeb vector field

Since

$$\eta \wedge (d\eta)^n = vol_M,$$

there is an isomorphism of C^∞-modules

$$\chi(M) \cong \Omega^{2n}(M),\; X \to i_X vol_M.$$

Hence there is a unique $\xi \in \chi(M)$ such that $i_\xi(\eta \wedge (d\eta)^n) = (d\eta)^n$. Contracting twice:

$$0 = i_\xi i_\xi(\eta \wedge (d\eta)^n = i_\xi (d\eta)^n \Rightarrow i_\xi d\eta = 0,$$

because rank $d\eta = 2n$. Also

$$i_\xi(\eta \wedge (d\eta)^n) = \eta(\xi)(d\eta)^n - \eta \wedge i_\xi(d\eta)^n = (d\eta)^n \Rightarrow \eta(\xi)(d\eta)^n = (d\eta)^n$$

hence $\eta(\xi) = 1$.

K-contact manifolds

We say that (M,η) is ***K-contact*** if there is an endomorphism Φ of TM such that the following conditions are satisfied:

1. $\Phi^2 = -Id + \xi \otimes \eta$, where ξ is the Reeb vector field of η;
2. the contact form η is compatible with Φ in the sense that

$$d\eta(\Phi X, \Phi Y) = d\eta(X, Y)$$

   for all X, Y and $d\eta(\Phi X, X) > 0$ for all nonzero $X \in \operatorname{Ker} \eta$;
3. The symmetric form

$$g(X, Y) = d\eta(\Phi(X), Y) + \eta(X)\eta(Y)$$

   is a Riemannian metric on M;
4. the Reeb field of η is a Killing vector field with respect to the Riemannian metric g, ($\mathcal{L}_\xi g = 0$) and $\xi \perp \mathcal{F}$.

Geometric structures involved:

- Endomorphism Φ defines a complex structure on the distribution $\operatorname{Ker}\eta$: if $X \in \mathcal{F}$, then $\Phi^2(X) = -\mathrm{id}(X) + \eta(X)\xi = -X$. Thus, $\mathcal{F}$ is a complex vector bundle.
- Φ is compatible with the "symplectic structure" $d\eta$: $d\eta(\Phi(X), \Phi(Y)) = d\eta(X, Y)$;
- $g|_{\mathcal{F}\times\mathcal{F}}$ is compatible with a symplectic structure $d\eta$: $g(X, Y) = d\eta(\Phi(X), Y)$, $d\eta(\Phi(X), X) > 0$.
- Φ is orthogonal with respect to the associated metric $g = d\eta \circ (\Phi \otimes Id)$.
- By definition, the Reeb field ξ is g-orthogonal to $\operatorname{Ker}\eta$.

Circle action on K-contact manifold

Since ξ is Killing, there is a one-parametric subgroup of $\mathrm{Iso}(M, g)$, since M is compact, there is a circle action $S^1 \times M \to M$ determined by ξ, and since $||\xi|| = \mathit{const}$ ($\eta(\xi) = 1$), it has no fixed points (and, hence, finite isotropy).

Proposition

Any compact K-contact manifold admits a circle action with finite isotropy.

K-contact and symplectic cones

For a contact manifold (M, η) define the ***metric cone*** or ***the symplectization*** as

$$\mathcal{C}(M) = (M \times \mathbb{R}^{>0}, t^2 g + dt^2).$$

Given a K-contact manifold (M, η, Φ, g), the almost complex structure I on $\mathcal{C}(M)$ is defined by:

1. $I(X) = \Phi(X)$ on $\operatorname{Ker} \eta$;
2. $I(\xi) = t\frac{\partial}{\partial t}$, $I(t\frac{\partial}{\partial t}) = -\xi$.

A characterization of K-contact manifolds via symplectic cones:

Proposition

A quadruple (M, η, Φ, g) is K-contact if and only if $\mathcal{L}_\xi I = 0$.

Proof.

$$\mathcal{L}_\xi I = \mathcal{L}_\xi \Phi = 0 \Leftrightarrow \mathcal{L}_\xi g = 0.$$

Examples of K-contact manifolds: Boothby-Wang

Let (B, ω_B) be symplectic, $[\omega] \in H^2(B, \mathbb{Z})$, and let

$$S^1 \to M \to B$$

be a principal circle bundle determined by $[\omega_B] \in H^2(B, \mathbb{Z})$. There is a connection with a connection form

$$\theta \in \Omega^1(M, L(S^1)) = \Omega(M),\ d\theta = \pi^*\omega_B,\ \pi : M \to B.$$

Since $\mathcal{H} = \mathrm{Ker}\theta$ is a horizontal distribution, there is a metric g on $\mathcal{H}$ lifted from B (and extended onto M) via $TM = TS^1 \oplus \mathcal{H}$. One can lift $g(X, Y) = \omega_B(JX, Y)$, that is, a Riemannian metric on B compatible with ω_B.

Proposition

Any Boothby-Wang S^1-bundle carries a K-contact structure.

Simplest: the Hopf bundle

$$S^1 \to S^{2n+1} \to \mathbb{C}P^n$$

More examples: homogeneous spaces

Let $B_\xi = G/G_\xi \subset \mathfrak{g}^*$ be a co-adjoint orbit of a compact G:

$$G/G_\xi = \{(Ad\,g)^*(\xi) \mid g \in G\},$$
$$(Ad\,g)^*(\xi)(X) := \xi(Ad\,g^{-1}(X)\ X \in \mathfrak{g}.$$

Co-adjoint orbits are symplectic with respect to the invariant 2-form on the homogeneous space G/G_ξ.

$$\omega_\xi(X, Y) = \xi([X, Y]),\ \mathrm{Ker}\,\omega_\xi = \mathfrak{g}_\xi,\ T_\xi(G/G_\xi) \cong \mathfrak{g}/\mathfrak{g}_\xi.$$

Consider

$$\mathfrak{l}_\xi = \{X \in \mathfrak{g}_\xi \,|\, \xi(X) = 0\}.$$

If there is a *closed* subgroup L_ξ corresponding to $\mathfrak{l}_\xi$, then

$$G_\xi/L_\xi \to G/L_\xi \to G/G_\xi$$

is Boothby-Wang.

K-Contact homogeneous spaces, cntd.

Let $T = Z_G(\xi)$, ξ is regular. Then T is a maximal torus in G, $G/G_\xi = G/T$ is a flag manifold. If $\chi \in X(T)$ is a non-trivial character such that $d\chi = \xi$, then $(\mathrm{Ker}\,\chi)_0 = T' = L_\xi$ is a closed subtorus in T, and one obtains the Boothby-Wang fiber bundle

$$T/T' = S^1 \to G/T' \to G/T'.$$

More K-contact - orbifolds

Orbifold charts on a paracompact Hausdorff space

A triple $(\hat{U}, \Gamma, \varphi)$, where

- $\hat{U} \subset \mathbb{R}^n$ is connected and open, $\{0\} \in \hat{U}$,
- $\Gamma \subset \operatorname{Diff} \hat{U}$ is a finite group,
- $\varphi : \hat{U} \to U \subset X$ such that $\varphi \circ \gamma = \varphi$, $\forall \gamma \in \Gamma$,
- $\hat{U}/\Gamma \cong U$ (homeo).

Injection of charts

$(\hat{U}, \Gamma, \varphi) \hookrightarrow (\hat{U}', \Gamma', \varphi')$ is a smooth embedding $\lambda : \hat{U} \hookrightarrow \hat{U}'$ such that $\varphi' \circ \lambda = \varphi$.

Orbifold atlas

An orbifold atlas on X is a collection $\mathcal{U} = \{(\hat{U}_i, \Gamma_i, \varphi_i)\}$ such that

1. $X = \cup_i \varphi_i(\hat{U}_i)$;
2. compatiblity: given two charts $(\hat{U}_i, \Gamma_i, \varphi_i), (\hat{U}_j, \Gamma_j, \varphi_j)$ with $U_i = \varphi_i(\hat{U}_i)$ and $U_j = \varphi_j(\hat{U}_j)$ and a point $x \in U_i \cap U_j$ there is a neighbourhood U_k of x and a chart $(\hat{U}_k, \Gamma_k, \varphi_k)$ with chart injections

$$\lambda_{ik} : (\hat{U}_k, \Gamma_k, \varphi_k) \hookrightarrow (\hat{U}_i, \Gamma_i, \varphi_i),\ \lambda_{jk} : (\hat{U}_k, \Gamma_k, \varphi_k) \hookrightarrow (\hat{U}_j, \Gamma_j, \varphi_j)$$

Orbifolds

$(X, \mathcal{U})$ is an orbifold.

regular and singular point of orbifolds

$\forall p \in \varphi^{-1}(x)$ the isotropy Γ_p depends only on x (up to conjugation), hence $\Gamma_x \subset \Gamma$.

$x \in X$ is singular, if $\Gamma_x \neq \{e\}$, regular otherwise.

$$X = X^{reg} \cup X^{sing}$$

X^{reg} is open and dense in X.

Basic example of an orbifold

Let a topological group Γ act on a compact manifold M in a way that for any $p \in M$ there is a neighbourhood U of p such that

$$\{\gamma \in \Gamma \mid U \cap \gamma \cdot U \neq \emptyset\}$$

is finite for any $\gamma \neq e$.

Proposition

M/Γ is an orbifold.

Proof. For any $p \in M$ the isotropy Γ_p is finite. Also, there is $\tilde{U}_p$ such that $\gamma\tilde{U}_p \cap \tilde{U}_p = \emptyset$, if $\gamma \notin \Gamma_p$. One may assume $\gamma\tilde{U}_p = \tilde{U}_p$ for $\gamma \in \Gamma_p$. Then $\varphi : \tilde{U}_p \to \tilde{U}_p/\Gamma_p$ is a local homeo,

$$\varphi \circ \gamma = \varphi.$$

Then $\mathcal{U} = \{(\tilde{U}_p, \varphi, \Gamma_p)\}$ is an orbifold atlas for M/Γ.

Important particular example - weighted projective space

$\mathbb{C}^{n+1}$ with the weighted $\mathbb{C}^*$-action:

$$(z_0, ..., z_n) \to (\lambda^{w_0} z_0, ..., \lambda^{w_n} z_n)$$

for $w = (w_0, ..., w_n) \in \mathbb{Z}^{n+1} > 0$. Then

$$\mathbb{C}P^n(w) := \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*.$$

Clearly, the action has the property $U \cap gU$ is finite. Also, an orbifold atlas can be seen explicitly:

$$\tilde{U}_i = \{(z_0, ..., z_n) \,|\, z_i = 1\}, \Gamma_i = \{\lambda \in S^1, \lambda^{w_i} = 1\}.$$

$$U_i = \tilde{U}_i/\Gamma_i \cong \{[z_0, ..., z_n] \in \mathbb{C}P(w) \,|\, z_i \neq 0\}.$$

Geometry of orbifolds: orbi-bundles (locally)

- $\mathcal{U}$ an orbifold atlas, $\tilde{U}$ an orbifold chart;
- a fiber bundle $F \to B_{\tilde{U}} \to \tilde{U}$ with a (manifold) fiber F and a structure group G;
- monoomorphisms $h_{\tilde{U}_i} : \Gamma_i \to G$ with the properties:

  a) if $b \in F$ over some $\tilde{x}_i \in \tilde{U}_i \Rightarrow \forall \gamma \in \Gamma_i$, $bh_{\tilde{U}_i}(\gamma) \in F_y$, where $y = \gamma^{-1}\tilde{x}_i$;

  b) If $\lambda_{ij} : \tilde{U}_i \to \tilde{U}_j$ is an injection of orbi-charts, then there is a bundle map $\lambda_{ji}^* : B_{\tilde{U}_j}|_{\lambda_{ji}(\tilde{U}_i)} \to B_{\tilde{U}_i}$, satisfying the following: if $\gamma \in \Gamma_i$, and $\gamma' \in \Gamma_j$ is the unique element such that $\lambda_{ji} \circ \gamma = \gamma' \circ \lambda_{ji}$, then

$$h_{\tilde{U}_i}(\gamma) \circ \lambda_{ji}^* = \lambda_{ji}^* \circ h_{\tilde{U}_j}(\gamma'),$$

  and if λ_{kj} is another such injection, then $(\lambda_{kj} \circ \lambda_{ji})^* = (\lambda_{ji}^* \circ \lambda_{kj}^*$.

The total space of an orbi-bundle

The gluing is a modification of the bundle construction.

- Take $B_{\tilde{U}_i}$ and define $\varphi_i^* : B_{\tilde{U}_i} \to B_{\tilde{U}_i}/\Gamma_i^*$;
- Γ_i^* are defined as follows: $B_{\tilde{U}_i} \cong \tilde{U}_i \times F$, the action of Γ_i extends onto $\tilde{U}_i \times F$:

$$(\tilde{x}_i, b) \to (\gamma^{-1}\tilde{x}_i, bh_{\tilde{U}_i}(\gamma),$$

  Γ_i^* are stabilizers of (x_i, b). Thus,

$$B_{\tilde{U}_i}, \Gamma^*$$

  constitute a local uniformization system
- the total space E is obtained by gluing together $B_{\tilde{U}_i}/\Gamma_i^*$:

$$[(\tilde{x}_i, b)] \simeq [(\tilde{x}_j, c)], \text{ iff } bh_{\tilde{U}_i}(\gamma) = c, \gamma^{-1}x_i = x_j.$$

Tangent bundle and the rest...

If $F = \mathbb{R}^n$, and $B_{\tilde{U}} = T\tilde{U}$, one obtains the tangent bundle TX, hence:

- sections of tangent bundles, its symmetric and exterior powers yield vector fields, Riemannian metrics, differential forms,...
- complex orbifolds,
- symplectic orbifolds,
- hermitian metrics on complex orbifolds
- Kaehler orbifolds.

Kaehler orbifolds

Let X be a complex orbifold with a hermitian metric g. Consider the skew 2-form $\omega_g(X, Y) := g(X, JY)$ for a complex structure J on X. X is Kaehler, if $d\omega_g = 0$.

Topological characterization of K-contact manifolds

Theorem

A closed odd-dimensional manifold M admits a structure of a K-contact manifold, if and only if it is a total space of a principal circle orbi-bundle

$$S^1 \to M \to B$$

over a compact symplectic orbifold (B, ω_B).

Proof

quasi-regular contact structures

A K-contact structure on a compact manifold M is called *quasi-regular* if there is a positive integer δ satisfying the condition that each point of M has a foliated coordinate chart (U, t) with respect to ξ (the coordinate t is in the direction of ξ) such that each leaf for ξ passes through U at most δ times. If $\delta = 1$, then the Sasakian or K-contact structure is called *regular*.

Remark

M admits a regular K-contact structure if and only if M is a Boothby-Wang fiber bundle.

Remarks on proof in the quasi-regular case

If M admits a quasi-regular K-contact structure, it has a structure of an S^1-orbi-bundle over a symplectic orbifold (B, ω_B) (with cyclic quotient singularities) and Euler class $[\omega] \in H^2(M, \mathbb{Z})$, where $[\omega]$ is an orbifold symplectic form.

1. ξ is Killing, $\Rightarrow$ leaves of the corresponding foliation $\mathcal{F}$ are geodesics, and it is a Riemannian foliation with compact leaves;
2. Since the leaves are geodesics, they are orbits of a locally free S^1-action $\Rightarrow \pi : M \to M/\mathcal{F}_\xi$ is a principal circle orbibundle (Wadsley);
3. the subbundle $\mathcal{F} = \mathrm{Ker}\,\eta$ is S^1-invariant $(\xi \perp \mathcal{F}) \Rightarrow \mathcal{F}$ is a **connection**;
4. $d\eta$ is a symplectic form on $\mathcal{F} \Rightarrow$ there is Φ;
5. A symplectic form is obtained from $d\eta$ "via a connection".

general case

If a compact manifold M admits a K-contact structure, it admits a quasi-regular K-contact structure.

Sasakian manifolds

A K-contact manifold is called ***Sasakian***, if the almost complex structure I is integrable, hence defines a dilatation-invariant complex structure on $\mathcal{C}(M)$, endowing $\mathcal{C}(M)$ with a Kähler structure.

Example

Still, the total space of the Hopf bundle

$$S^1 \to S^{2n+1} \to \mathbb{C}P^n$$

is an example of a Sasakian manifold.

Relations between K-contact and Sasakian

- Any Sasakian manifold is K-contact by definition,
- The difference is that one adds to the definition of the quadruple (M, η, g, Φ) an extra assumption that $I : TC(M) \to TC(M)$ the integrability condition:

$$N_I(X, Y) = [IX, IY] - I[IX, Y] - I[X, IY] - [X, Y] = 0,$$

hence the symplectic cone $TC(M)$ becomes Kaehler.
- If M admits a Sasakian structure, M is a total space of an S^1-orbibundle over a Kaehler orbifold (B, ω_B).

K-contact vs. Sasakian/symplectic vs. Kaehler

Do there exist compact K-contact manifolds which do not carry Sasakian structures?

Research program

Topological obstructions to the existence of K-contact/Sasakian structures on a compact manifold M of dimension $2n+1$:

1. the evenness of the p-th Betti number for p odd with $1 \leq p \leq n$, of a Sasakian manifold,
2. some torsion obstructions in dimension 5 discovered by Kollár,
3. the fundamental groups of Sasakian manifolds are special,
4. the cohomology algebra of a Sasakian manifold satisfies (a version of) the hard Lefschetz property,
5. formality properties of the rational homotopy type.

Boyer and Galicki program

Study topological properties of K-contact and Sasakian manifolds, in particular, understand the question of “K-contact vs. Sasakian”.

Some recent results

Based on:

- C. Boyer, K. Galicki, *Sasakian Geometry*, Oxford, 2009
- B. Capeletti-Montano, A. De Nicola, I. Yudin, *Hard Lefschetz theorem for Sasakian manifolds*, ArXiv: 1306.2896
- B. Hajduk, A. Tralle, *On simply connected K-contact non-sasakian manifolds*, J. Fixed Point Theory Appl. 16(2014), 229-241
- V. Muñoz, A. Tralle, *Simply connected K-contact and Sasakian manifolds of dimension 7*, Math. Zeitschrift, to appear
- I. Biswas, M. Fernández, V. Muñoz, A. Tralle, *On formality of Sasakian manifolds*, J. Topol., to appear
- A. Tralle, J. Oprea, *Symplectic manifolds with no Kaehler structure*, Springer, 1997

THEOREM 1

There exist simply connected K-contact manifolds which do not carry any Sasakian structure in dimensions ≥ 7

THEOREM 2

The higher order Massey products of a compact Sasakian manifold vanish.

THEOREM 3

Let Γ *be an irreducible lattice in a semisimple real Lie group G of real rank at least 2 with no co-compact factors and with trivial center. If* Γ *is Sasakian, then it must be isomorphic to the group* $\pi_1^{orb}(M)$ *of some Kaehler orbifold. Moreover,* Γ *cannot be a co-compact arithmetic lattice in* $SO(1, n)$, $n > 2$, *or* $F_{4(20)}$, *or a simple non-hermitian Lie group of non-compact type with real rank at least 20.*

THEOREM 4

Any finitely presented group is K-contact

A challenging problem

Do there exist simply connected closed K-contact 5-manifolds ("Smale-Barden manifolds") which do not carry Sasakian structures?

A proof for dimensions ≥ 9

Our construction is based on:

Betti numbers of Sasakian manifolds

If M is a closed Sasakian manifold of dimension $2n+1$, then for any odd $p \leq n$ the Betti numbers b_p are even.

Fiber bundles with fiberwise K-contact structures

There exist associated fiber bundles with structure group G (a Lie group)

$$F \to P \times_G F \to M$$

with

1. simply connected base M and fiber F,
2. K-contact fiberwise structure and odd $b_p(P \times_G F)$.

Why $b_p(M)$ are even for Sasakian M?

Theorem

Any Sasakian manifold M is the total space of a (non-trivial) principal circle orbibundle over a Kähler orbifold B.

Theorem, Wang-Zaffran

Kähler orbifolds satisfy the hard Lefschetz property.

Hard Lefschetz

Let $[\omega] \in H^2(M)$, define $L_\omega^k : H^{n-k}(M) \to H^{n+k}(M)$:

$$L_\omega^k(\beta) = [\omega^k \wedge \beta].$$

M satisfies the *Hard lefshetz property*, if L_ω^k are isomorphisms for $k \leq n$.

Evennes - explanation

Gysin sequence for orbibundles $\pi : M \to B$

$$... \to H^{p-2}(B) \stackrel{L_\omega}{\to} H^p(B) \stackrel{\pi^*}{\to} H^p(M) \to H^{p-1}(B) \stackrel{L_\omega}{\to} H^{p+1}(B) \to \dots .$$

By assumption $L_\omega^i = L_\omega^{i-1} L_\omega$ is an isomorphism, ($i = n + p - 1$), thus for $i \leq n$ the linear map $L_\omega : H^{p-1}(B) \to H^{p+1}(B)$ is a monomorphism.
The Gysin sequence gives that π^* is onto and
$H^p(M) = H^p(B)/H^{p-2}(B)$ is even dimensional for any odd $p \leq n$ (because B is Kaehler $\Rightarrow b_p(B)$ are even for odd p).

A fiberwise construction of K-contact manifolds I

Fatness

Let $G \to P \to B$ be a principal bundle with a connection. Let θ and Θ be the connection one-form and the curvature form of the connection, respectively. Both forms have values in the Lie algebra $\mathfrak{g}$ of the group G. Denote the pairing between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ by $\langle\, , \rangle$. By definition, a vector $u \in \mathfrak{g}^*$ is ***fat***, if the two–form

$$(X, Y) \to \langle \Theta(X, Y), u \rangle$$

is nondegenerate for all *horizontal* vectors X, Y.

A fiberwise construction of K-contact manifolds II

Contact moment map

Let (M, η) be a contact co-oriented manifold endowed with a contact action of a Lie group G. Define a ***contact moment map*** by the formula

$$\mu_\eta : M \to \mathfrak{g}^*, \ \langle \mu_\eta(x), X \rangle = \eta_x(X^*_x)$$

for any $x \in M$ and any $X \in \mathfrak{g}$. We denote by X^* the fundamental vector field on M generated by $X \in \mathfrak{g}$.

A fiberwise construction of K-contact manifolds III

Theorem (Lerman)

Let there be given a contact G-manifold (F, η) with the contact moment map ν. Assume that

$$G \to P \to M$$

is a principal fiber bundle endowed with a connection such that the image $\nu(F) \subset \mathfrak{g}^*$ consists of fat vectors. Then there exists a fiberwise contact structure on the total space of the associated bundle

$$F \to P \times_G F \to M.$$

If the fiber (F, η) is K-contact and G preserves the K-contact structure, then the total space of the associated bundle is also K-contact.

Final construction

Theorem (B.Hajduk, A.T.)

Let (B, ω) be any compact simply connected symplectic manifold such that $b_3(B)$ is odd. If

$$S^1 \to P \to B$$

is a Boothby-Wang fibration with the Euler class equal to $[\omega]$, then the total space of the fiber bundle

$$S^3 \to P \times_{S^1} S^3 \to B$$

associated to $P \to B$ by the Hopf action of S^1 on S^3 admits a K-contact structure, but no Sasakian structure.

$b_3(M) = b_3(B)$ for $M = P \times_{S^1} S^3$, $\dim B \geq 6 \Rightarrow \dim M \geq 9$.

Dimension 7

Theorem (V. Muñoz, A.T.)

There exist 7-dimensional compact simply connected K-contact manifolds which do not admit a Sasakian structure.

Sketch of proof: Gompf-Cavalcanti manifold

Let (M, ω) be a symplectic manifold of dimension $2n$. For every $0 \leq k \leq n$, we define the Lefschetz map as $L_\omega : H^{n-k}(M) \to H^{n+k}(M)$, $L_\omega([\beta]) = [\beta \wedge \omega^{n-k}]$. We say that M satisfies the hard Lefschetz property if L_ω is an isomorphism for every $0 \leq k \leq n$.

Proposition

There exists a simply connected 6-dimensional symplectic manifold (M, ω) ("Gompf-Cavalcanti manifold") such that $\dim \ker(L_\omega : H^2(M) \to H^4(M))$ is odd.

Proposition (Gompf)

There exists a simply connected 6-dimensional symplectic manifold (M, ω) which does not satisfy the hard Lefschetz property, that is, the Lefschetz map $L_\omega : H^2(M) \to H^4(M)$ is not an isomorphism.

Therefore, if $\dim \operatorname{Ker}(L_\omega)$ is odd, there is nothing to prove.

Assumption $\dim \operatorname{Ker} L_\omega$ is even

Take a cohomology class $a \in \operatorname{Ker} L_\omega \subset H^2(M)$.

Proposition (Cavalcanti)

Given a symplectic manifold (M, ω) as above satisfying that there exists a symplectic surface $S \hookrightarrow M$ with $\langle a, [S] \rangle \neq 0$, then there is another 6-dimensional symplectic manifold (M', ω') (the symplectic blow-up of M along S) satisfying

$$\dim \ker(L_{\omega'} : H^2(M') \to H^2(M')) = \dim \ker(L_\omega : H^2(M) \to H^2(M)) - 1.$$

Clearly $\pi_1(M') \cong \pi_1(M)$. Hence, it remains to prove that there exists $S \hookrightarrow M$.

Existence of a symplectic surface $S \hookrightarrow M$

- The cohomology class $a \neq 0 \Rightarrow$ there is some $b \in H^4(M, \mathbb{Z})$ such that $a \cup b \neq 0$;
- (A homotopic argument) There exists a rank 2 complex vector bundle E such that

$$c_1(E) = 0, c_2(E) = 2b.$$

- Let $L \to M$ be the line bundle with $c_1(L) = [\omega]$, and apply the asymptoticall holomorphic technique of Donaldson: for a suitably large $k >> 0$ thee exists a section

$$M \to E \otimes L^{\otimes k}$$

  whose zero locus S is a symplectic surface (Auroux);
- the dual to $[S]$ is $c_2(E \otimes L^{\otimes k} = c_2(E) + 2kc_1(L) = 2b + 2k[\omega]$;
- $\langle [S], a \rangle = a \cup (2b + 2k[\omega]) = 2a \cup b \neq 0$

Completion: Boothby-Wang over Gompf-Cavalcanti

Theorem

The total space E of the Boothby-Wang fibration

$$S^1 \to E \to M$$

over the Gompf-Cavalcanti manifold M is a simply connected K-contact non-Sasakian manifold of dimension 7.

1. E is simply-connected: use Serre spectral to get $H_1(E) = 0$, and long homotopy to get $\pi_1(E) = \{1\}$;
2. use Gysin

$$H^1(M) = 0 \xrightarrow{\wedge\omega} H^3(M) \longrightarrow H^3(E) \longrightarrow H^2(M) \xrightarrow{\wedge\omega} H^4(M).$$

3. thus

$$b^3(E) = b^3(M) + \dim(\ker L_\omega : H^2(M) \to H^4(M)).$$

Dimension 5?

Kodaira-Baily theorem

A Hodge orbifold is a projective algebraic variety.

Thus, one of the approaches is to find some 4-dimensional *symplectic* orbifold which has the property that the total space of the Boothby-Wang fibration over it cannot fiber over a projective complex surface...
No good conjecture at hand...
The dimension is too small to use (only) homotopic methods...

Massey products

Our aim is to explain the role of Theorem 2.
We repeat the result:

Theorem 2

The higher order Massey products of a compact Sasakian manifold vanish. Hence these do obstruct Sasakian structures.

The definition of triple Massey products

Let $(\mathcal{A}, d)$ be a DGA (in particular, it can be the de Rham complex of differential forms on a differentiable manifold). Suppose that there are cohomology classes $[a_i] \in H^{p_i}(\mathcal{A})$, $p_i > 0$, $1 \leq i \leq 3$, such that $a_1 \cdot a_2$ and $a_2 \cdot a_3$ are exact. Write $a_1 \cdot a_2 = da_{1,2}$ and $a_2 \cdot a_3 = da_{2,3}$. The *(triple) Massey product* of the classes $[a_i]$ is defined to be

$$\langle [a_1], [a_2], [a_3] \rangle = [a_1 \cdot a_{2,3} + (-1)^{p_1+1} a_{1,2} \cdot a_3] \in$$

$$\in \frac{H^{p_1+p_2+p_3-1}(\mathcal{A})}{[a_1] \cdot H^{p_2+p_3-1}(\mathcal{A}) + [a_3] \cdot H^{p_1+p_2-1}(\mathcal{A})}.$$

Higher order Massey products

Given

$$[a_i] \in H^*(\mathcal{A}), \;\; 1 \leq i \leq t, \;\; t \geq 3,$$

the Massey product $\langle [a_1], [a_2], \cdots, [a_t] \rangle$, is defined if there are elements $a_{i,j}$ on $\mathcal{A}$, with $1 \leq i \leq j \leq t$ and $(i,j) \neq (1,t)$, such that

$$\begin{aligned} a_{i,i} &= a_i \,, \\ d\, a_{i,j} &= \sum_{k=i}^{j-1} (-1)^{|a_{i,k}|} a_{i,k} \cdot a_{k+1,j} \,. \end{aligned} \tag{1}$$

Then the *Massey product* is the set of cohomology classes

$$\langle [a_1], [a_2], \cdots, [a_t] \rangle$$

$$= \left\{ \left[\sum_{k=1}^{t-1} (-1)^{|a_{1,k}|} a_{1,k} \cdot a_{k+1,t} \right] \;\middle|\; a_{i,j} \text{ as in (1)} \right\} \subset H^{|a_1|+\ldots+|a_t|-(t-2)}(\mathcal{A}) \,.$$

The role of Massey products: minimal models

A DGA $(\mathcal{A}, d)$ is *minimal* if:

1. $\mathcal{A}$ is free as an algebra, that is, $\mathcal{A}$ is the free algebra $\bigwedge V$ over a graded vector space $V = \bigoplus_i V^i$, and
2. there is a collection of generators $\{a_\tau\}_{\tau \in I}$ indexed by some well ordered set I, such that $|a_\mu| \leq |a_\tau|$ if $\mu < \tau$ and each da_τ is expressed in terms of preceding a_μ, $\mu < \tau$.

We shall say that $(\bigwedge V, d)$ is a *minimal model* of the differential graded commutative algebra $(\mathcal{A}, d)$ if $(\bigwedge V, d)$ is minimal and there exists a morphism of differential graded algebras

$$\rho\colon (\bigwedge V, d) \longrightarrow (\mathcal{A}, d)$$

inducing an isomorphism $\rho^*\colon H^*(\bigwedge V) \xrightarrow{\sim} H^*(\mathcal{A})$ of cohomologies. A minimal algebra $(\bigwedge V, d)$ is called *formal* if there exists a morphism of differential algebras $\psi\colon (\bigwedge V, d) \longrightarrow (H^*(\bigwedge V), 0)$ inducing the identity map on cohomology.

Massey products obstruct formality

Theorem (Deligne-Griffiths-Morgan-Sullivan)

A DGA which has a non-zero Massey product is not formal.

Theorem (Deligne-Griffiths-Morgan-Sullivan)

Compact Kaehler manifolds are formal (and, hence, all their Massey products vanish).

Quotations

J.-P. Bourgignon

"... several dominant figures of the mathematical scene of the XX century have, step after step along a 50 year period, transformed the subject [Kaehler geometry] into a major area of mathematics that has influenced the evolution of the discipline much further than could have conceivable been anticipated by anyone. "

Krzysztof Galicki

"Sasaki seemed to have had both the necessary intuition and a broad vision in understanding what is and what is not of true importance..."

Shigeo Sasaki (1912-1987)

- born in 1912 in Yamagate prefecture to a farmer's family, brought up by his uncle who was a superior of a Buddist Temple
- studied in Tohoku Imperial University (1932-1935)
- since 1935 worked in differential geometry under the guidance of prof. Kubota, Ph. D. in 1943
- in 1946 appointed to the vacant chair after Kubota's retirement
- 1962 - introduced the notion of "normal metric contact structure' which is equivalent to "Sasakian structure";
- Major works in 1962-1967, notes in Japanese, creation of a subfield in Riemannian geometry
- professorship in Tohoku University until 1976, visited to Princeton in 1952-1954, worked with Veblen, Morse and Chern
- His biography is not well known, even his obituary appeared only in a local newspaper