Topology of torus actions and combinatorics

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Study nice spaces with torus actions from topological viewpoint and find relations to other fields in mathematics

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[2] Toric Topology, Mathematical Surveys and Monographs 204, AMS, 2015 July (518 pp)

Content of this talk

§1. **Brief review of toric geometry**

§2. **Cohomology of toric manifolds**

§3. **Counting lattice points and face numbers**

§4. **Moment-angle manifolds**

Toric variety

Toric geometry has been established in 1970's by Demazure, Miyake-Oda, Mumford, ...

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Example

$(g_1, \dots, g_n) \in (\mathbb{C}^*)^n$

(1) $(\mathbb{C}^*)^n \curvearrowright \mathbb{C}^n \quad (z_1, \dots, z_n) \mapsto (g_1 z_1, \dots, g_n z_n)$

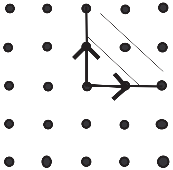
(2) $(\mathbb{C}^*)^n \curvearrowright \mathbb{C}P^n \quad [z_1, \dots, z_n, z_{n+1}] \mapsto [g_1 z_1, \dots, g_n z_n, z_{n+1}]$

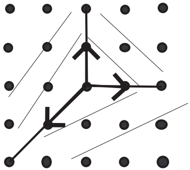
(3) $F_a := P(\mathbb{C} \oplus \mathcal{O}(a)) \to \mathbb{C}P^1$ $(a \in \mathbb{Z})$ Hirzebruch surface

Definition (Fan)

N: a free abelian group of rank n. A fan Δ in N is a collection of (rational strongly convex polyhedral) cones in $N \otimes \mathbb{R}$ s.t.

1. Each face of a cone in Δ is also a cone in Δ;
2. The intersection of two cones in Δ is a face of each.





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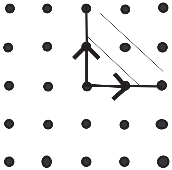

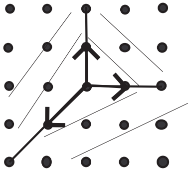

Fundamental theorem in toric geometry

$$\{\text{ toric varieties } X^n \curvearrowleft (\mathbb{C}^*)^n\} \quad \overset{1:1}{\Longleftrightarrow} \quad \{\text{fans in } \mathbb{R}^n\}$$

toric manifold $\Longrightarrow$ fan

$(\mathbb{C}^*)^n \curvearrowright X$ **: a toric manifold**

X_i $(i = 1, \ldots, m)$ **: invariant divisors (**$\mathrm{codim}_{\mathbb{C}} = 1$**)**

$X \setminus \cup_i X_i = (\mathbb{C}^*)^n$ **the dense orbit**

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Two data

- $K := \{I \subset [m] = \{1, \dots, m\} \mid \bigcap_{i \in I} X_i \neq \emptyset\}$ **simplicial complex**
- $v_i \in \mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n) = N$ **such that**
  1. $v_i(\mathbb{C}^*)$ **fixes** X_i,
  2. $v_i(g)_*(\xi) = g\xi$ **for** $\xi \in \nu(X_i)$ **(normal bundle)**

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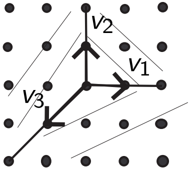
Span a cone by v_i's $(i \in I)$ whenever $I \in K \rightsquigarrow$ fan of X

Example

$(\mathbb{C}^*)^2 \curvearrowright \mathbb{C}P^2 = X \quad [z_1, z_2, z_3] \to [g_1 z_1, g_2 z_2, z_3]$

$X_i := \{z_i = 0\}$ $(i = 1, 2, 3)$ invariant divisors

- $K = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}\}$
- $v_1(g) = (g, 1),\ v_2(g) = (1, g),\ v_3(g) = (g^{-1}, g^{-1})\ \ (g \in \mathbb{C}^*)$



toric manifold $\Longleftarrow$ fan

Given $\Delta = (K, \{v_i\}_{i=1}^m)$ **(complete non-singular fan) where** K **is a simplicial cpx on** $[m]$ **and** $v_i \in \mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n) = N$**.**

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- $U(K) = \mathbb{C}^m \backslash \bigcup_{J \notin K} Z_J$ **where**

$$Z_J = \{z_{j_1} = \cdots = z_{j_k} = 0\} \quad (J = \{j_1, \ldots, j_k\} \subset [m])$$

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$$\lambda(h_1, \ldots, h_m) = v_1(h_1) \cdots v_m(h_m)$$

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Then

$$X(\Delta) := U(K)/\ker\lambda \quad \textbf{toric manifold}$$

Remark

(1) A toric manifold has local charts s.t. transition maps are Laurent monomials

$$(z_1, \dots, z_n) \in \mathbb{C}^n - - \longrightarrow (\prod_{j=1}^{n} z_j^{a_{1j}}, \dots, \prod_{j=1}^{n} z_j^{a_{nj}}) \in \mathbb{C}^n$$

where $(a_{ij}) \in \mathrm{GL}(n, \mathbb{Z})$.

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(3) Real toric manifold = the set of real points in a toric manifold. $\mathbb{R}P^n$ is an example. Real toric mfds are often aspherical mfds.

Betti number of toric manifold

- $(\mathbb{C}^*)^n \curvearrowright X$ **toric manifold**

$$q\colon X \to X/(S^1)^n = Q$$ **(often simple polytope)**

Morse theory applied to Q **induces even dimensional cell decomposition of** X**. Hence** $H^{odd}(X) = 0$**.**

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- $P = Q^*$ **simplicial polytope of** $\dim_{\mathbb{R}} = n$**.**

$f_i := \sharp$ **of i-dim faces of $P = \sharp$ of $(n-i-1)$-dim faces of Q**

$$(f_0, f_1, \dots, f_{n-1})\text{: } f\text{-vector}$$

Then the h-vector $(h_0, h_1, \dots, h_n)$ is defined by

$$(t-1)^n + f_0(t-1)^{n-1} + \cdots + f_{n-1} = h_0 t^n + h_1 t^{n-1} + \cdots + h_n$$

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Lemma (a bridge between topology and combinatorics)

$b_{2i}(X) = h_i \ (0 \le i \le n)$

Equivariant cohomology

$T = (S^1)^n \subset (\mathbb{C}^*)^n \curvearrowright X$: **toric manifold,**
$\Delta(X) = (K, \{v_i\}_{i=1}^m)$ **the fan associated to** X

$$H_T^*(X) := H^*(ET \times_T X)$$

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X_i $(i = 1, \dots, m)$ **: invariant divisors**

$$X_i \quad \overset{P.D.}{\longleftrightarrow} \quad \tau_i \in H_T^2(X)$$

For $I \subset \{1, \dots, m\}$

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Fact 1. $H^*_T(X)=\mathbb{Z}[\tau_1,\dots,\tau_m]\Big/\Big(\prod_{i\in I}\tau_i \mid I\notin K\Big)$ **as rings**

We have a fibration $X \longrightarrow ET \times_T X \xrightarrow{\pi} BT$**.**
Since $H^{odd}(X) = 0$ **and** $H^*(BT)$ **is generated by** $H^2(BT)$**,**

$$H^*(X) = H^*_T(X)/\big(\pi^* H^2(BT)\big)$$

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Fact 2. $\exists_1\ v_i \in H_2(BT)$ $(i = 1, \dots, m)$ **s.t.**

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Theorem (Danilov-Jurkiewicz)

$$H^*(X) = \mathbb{Z}[\mu_1, \ldots, \mu_m]/\mathcal{I}$$

where $\mathcal{I}$ *is the ideal generated by all*

1. $\prod_{i \in I} \mu_i$ *for* $I \notin K$;
2. $\sum_{i=1}^{m} \langle u, v_i \rangle \mu_i$ *for* $\forall u \in H^2(BT)$.

Topological analog of toric manifold

(1) Quasitoric manifold (Davis-Januskiewicz 1991)

$(S^1)^n \curvearrowright M^{2n}$ **closed smooth manifold with local charts iso. to the standard** $(S^1)^n \curvearrowright \mathbb{C}^n$ **and** $M/(S^1)^n$ **is a simple polytope**

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$$\{\text{quasitoric mfds}\} \subset \{\text{topological toric mfds}\} \subset \{\text{torus mfds}\}$$

Classification

Fundamental thm in toric geometry implies

$$X \cong Y \text{ as varieties} \iff \Delta(X) \cong \Delta(Y)$$

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Example (Hirzebruch surfaces)

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Cohomological rigidity problem for toric manifolds

Are two toric mfds X, Y diffeomorpic if $H^*(X) \cong H^*(Y)$?

True when $n = 1, 2$ (easy). The case $n = 3$ is open.

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Example (Bott manifolds, a good test case)

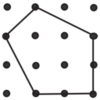
$B_n \to B_{n-1} \to \cdots \to B_2 \to B_1 \to B_0 = \{pt\}$ (Bott tower)
where $B_i = P(\mathbb{C} \oplus L_i) \to B_{i-1}$, L_i is a complex line bdle on B_{i-1}.

Moment map and polytope

To an ample T-line bundle $L \to X$ on a toric manifold

$$\exists \mu \colon X \to \mathrm{Lie}(T)^* \supset \mathrm{Hom}(T, S^1) = \mathbb{Z}^n$$

and $\mu(X)$ is a lattice convex polytope.



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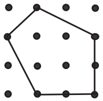

Theorem (well-known)

$$RR^T(X,L) = H^0(X;L) = \sum_{u \in \mu(X)} t^u \in R(T)$$

$$RR(X,L) = \dim_{\mathbb{C}} H^0(X;L) = \sharp(\mu(X)).$$

To an arbitrary T-line bundle $L \to M^{2n}$ on a torus mfd M, $RR^T(M, L) \in R(T)$ is still defined, and

$$\exists \mu \colon M \to \mathrm{Lie}(T)^* \supset \mathrm{Hom}(T, S^1) = \mathbb{Z}^n$$

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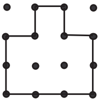

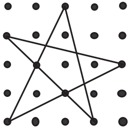

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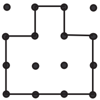
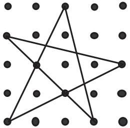




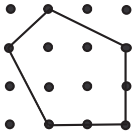
Theorem (Karshon-Tolman, Grossberg-Karshon, M)

$$RR^T(M, L) = \sum_{u \in \mu(M)} m(u) t^u \in R(T)$$

and $m(u) \in \mathbb{Z}$ can be described in terms of $\mu(M)$.

Theorem (Pick's formula, 1899)

If P is a lattice polygon, $\mathrm{Area}(P) = \#(\mathrm{Int}(P)) + \frac{1}{2}\#(\partial P) - 1$



$\mathrm{Area}(P) = 13/2$

$\#(\mathrm{Int}(P)) = 4$

$\#(\partial P) = 7$

Theorem (Pick's formula, 1899)

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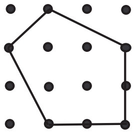
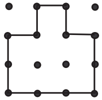


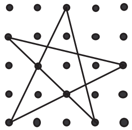
$\mathrm{Area}(P) = 13/2$

$\#(\mathrm{Int}(P)) = 4$

$\#(\partial P) = 7$

Pick's formula holds for the left and for the right with modification.





Face numbers of simplicial polytopes

P **:** **simplicial** n**-polytope**

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The h**-vector** $(h_0, h_1, \dots, h_n)$ **is defined as before:**

$$(t-1)^n + f_0(t-1)^{n-1} + \cdots + f_{n-1} = h_0 t^n + h_1 t^{n-1} + \cdots + h_n$$

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Theorem (*g*-theorem, Billera-Lee, Stanley, around 1980)

A vector of integers $(h_0, \dots, h_n)$ *with* $h_0 = 1$ *is the* h*-vector of a simplicial polytope* P *if and only if*

1. $h_i = h_{n-i}$ *for* $\forall i$ *(Dehn-Sommerville relations)*
2. $1 = h_0 \le h_1 \le \cdots \le h_{[n/2]}$
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Proof (Necessity).

$P \Longrightarrow$ **fan** $\Delta \Longrightarrow$ **projevtive toric manifold (orbifold)** X**.**

A key fact is $h_i(P) = b_{2i}(X)$**.**

(1) $\Longleftarrow$ **Poincaré duality (topology)**
(2) $\Longleftarrow$ **Hard Lefschetz theorem (algeberaic geometry)**
(3) $\Longleftarrow$ **Macaulay's theorem (commutative algebra)**

Face numbers of simplicial cell spheres

A simplicial cell $(n-1)$-sphere K is a CW complex whose cells are all simplices and whose realization is $\approx S^{n-1}$

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Note. $(1,0,1,0,1)$ **does not occur but** $(1,0,2,0,1)$ **does.**

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Idea of proof (Necessity) **Find a torus mfd M^{2n} s.t.**

$$h_i(\mathcal{P}) = b_{2i}(M)$$

As for (3), show $w(M) = \prod_{i=1}^{m}(1 + x_i) \quad (x_i \in H^2(M; \mathbb{Z}/2))$ **and use a fact that** $\chi(M) \equiv w_{2n}(M)[M] \pmod 2$**.**

Moment-angle manifolds Typical example:

$$S^{2n+1} \to S^{2n+1}/S^1 = \mathbb{C}P^n \to \mathbb{C}P^n/T^n = \Delta^n (= n\text{-}\textbf{simplex})$$

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For a simple n-polytope Q with m-facets, $\exists Z_Q$: a closed smooth mfd of $\dim_{\mathbb{R}} = m + n$ with T^m-action s.t.

1. Z_Q **is defined in $\mathbb{C}^m$ by quadrics**
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• Often $\exists$ subgroup $H(\cong T^{m-n})$ of T^m s.t. the H-action on Z_Q is free and Z_Q/H is a toric manifold.

• The topology of Z_Q depends on the combinatorial type of Q and is complicated in general.

(López de Medrano, Verjovsky, Bosio, Meersseman, Buchstaber, Panov, ...)

Concluding remark

A torus action on a space X is a useful tool to study X. For instance, if X is a closed manifold and $H^{odd}(X) = 0$, then

$$\iota^*\colon H^*_T(X) \to H^*_T(X^T) \quad \textbf{is injective}$$

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GKM theory (Goresky-Kottwiz-MacPherson) tells us the image of ι^*.

Imoportant manifolds or spaces such as Grasmann, flag and their subvarieties (Schubert, Springer, Peterson, Hessenberg etc.) are such examples and equivariant technique can be applied.

Congratulations on 370 Birthday of

Július Korbas
Krzysztof M. Pawalowski
Józef H. Przytycki
András Szücs
Pawel Traczyk
Robert Wolak