The Borel conjecture on aspherical manifolds

Tibor Macko

Universität Bonn and MI SAS Bratislava
www.math.uni-bonn.de/macko

Krakow, July 2015

The Borel conjecture on aspherical manifolds

Outline

- statement of the Borel conjecture
- s-cobordism theorem and algebraic K-theory
- surgery theory and L-theory
- assembly maps
- controlled topology

Aspherical manifolds I

Definition

A space X is called aspherical if for $i \neq 1$ we have

$$\pi_i(X) = \{1\}.$$

Corollary/Notation

If X is aspherical then $X \simeq K(G,1) \simeq BG$ where $G = \pi_1(X)$.

Remark

For every f.p. group G there exists $K(G,1) \simeq BG$.
Cell attaching, bar construction.

Question

Is $BG \simeq M^n$ where M^n is an n-dimensional closed (topological) manifold?

Aspherical manifolds II

Examples

- S^1, $T^n = (S^1)^n$, M and N aspherical then $M \times N$.
- F_g surface of genus $g \geq 1$.
- M^3 is aspherical iff irreducible and $\pi_1(M)$ is infinite.
- If M is a closed manifold with $\sec(M) \leq 0$ then by Cartan-Hadamard we have $\widetilde{M} \cong \mathbb{R}^n$ and hence $M \simeq K(G,1)$ for $G = \pi_1(M)$.
- $G \backslash L / K$ with L a Lie group, $K \subseteq L$ a max compact subgroup, $G \subseteq L$ a discrete torsion free subgroup
- exotic examples (Mike Davis) - for each $n \geq 4$ there exists a closed aspherical manifold M^n such that $\widetilde{M}$ is not homeomorphic to $\mathbb{R}^n$

Remark If M^n is a closed aspherical manifold then $\pi_1(M)$ is torsionfree.

The Borel conjecture I

The Borel conjecture

Let M and N be closed aspherical manifolds.
Then

$$M \cong N \quad \Leftrightarrow \quad M \simeq N \quad (\Leftrightarrow \quad \pi_1(M) \cong \pi_1(N) \,).$$

Remarks

- Poincaré conjecture $\quad M \simeq S^n \quad \Leftrightarrow \quad M \cong S^n.$
- $L(7;2,1) \simeq L(7;1,1)$ but $L(7;2,1) \not\cong L(7;1,1)$.
- Mostow rigidity theorem: if M, N are hyperbolic, then

$$M \cong_{\text{isometry}} N \quad \Leftrightarrow \quad M \simeq N.$$

The Borel conjecture II

Remarks

- Exotic spheres: there exists $\Sigma^n \simeq S^n$ such that $\Sigma^n \not\cong_{\mathrm{DIFF}} S^n$.
- The Borel conjecture is not true in the smooth category:

$$M \not\cong_{\mathrm{DIFF}} M \# \Sigma^n.$$

Historical remarks

- The Borel conjecture was conjectured by Borel in 1953.
- The Mostow rigidity was proved by Mostow in 1968.
- The Poincaré conjecture was proved for $n \geq 7$ by Smale in 1960.

The Borel conjecture III

Some special cases

It is true for:

- $M = S^1$
- $M = F_g$ with $g \geq 1$ by classification of surfaces.
- $M = T^n$, $n \geq 4$ (Farrell, Wall, Kirby, Siebenmann, Freedman, Quinn)
- M^3 by Thurston geometrization conjecture
- M is a closed non-positively curved manifold (Farrell-Jones)
- M is a closed manifold with $\pi_1(M)$ word-hyperbolic or CAT(0)-group (Bartels-Lueck).

The Borel conjecture IV

The 'existence' version

For G f.p. (finite) Poincaré duality group of dimension n there exists a closed manifold $M \simeq BG$.

Theorem (Bartels-Lueck-Weinberger)

Let G be a torsionfree hyperbolic group, let $n \geq 6$, and suppose that ∂G is homeomorphic to S^{n-1}. Then there exists a closed aspherical manifold M with $\pi_1(M) = G$ such that $\widetilde{M} \cong \mathbb{R}^n$.

The structure set I

Let X be a finite n-dim geometric Poincaré complex.

A manifold structure on X is $f\colon M \xrightarrow{\simeq} X$ with M an n-mfd.

Define

$$(f_0\colon M_0 \xrightarrow{\simeq} X) \sim (f_1\colon M_1 \xrightarrow{\simeq} X)$$

if there exists $h\colon M_0 \xrightarrow{\cong} M_1$ such that

$$f_1 \circ h \simeq f_0.$$

Definition:

The structure set of X is

$$\mathcal{S}^{\mathrm{TOP}}(X) := \{f\colon M \xrightarrow{\simeq} X\}/\sim$$

The Borel conjecture for aspherical manifold M.

The Borel conjecture holds for M if and only if $\mathcal{S}^{\mathrm{TOP}}(M) = \{1\}$.

The structure set II

Decorations.

Define

$$(f_0 \colon M_0 \xrightarrow{\simeq} X) \sim (f_1 \colon M_1 \xrightarrow{\simeq} X)$$

if there exists an h-cobordism $F \colon W \xrightarrow{\simeq} X \times [0,1]$ between f_0 and f_1.

$$\mathcal{S}^{\mathrm{TOP},h}(X) := \{f \colon M \xrightarrow{\simeq} X\}/\sim$$

Define

$$(f_0 \colon M_0 \xrightarrow{\simeq_s} X) \sim (f_1 \colon M_1 \xrightarrow{\simeq_s} X)$$

if there exists an s-cobordism $F \colon W \xrightarrow{\simeq_s} X \times [0,1]$ between f_0 and f_1.

$$\mathcal{S}^{\mathrm{TOP},s}(X) := \{f \colon M \xrightarrow{\simeq_s} X\}/\sim$$

The s-cobordism theorem I

An *h-cobordism* is a cobordism $(W; M_0, M_1)$ such that $M_i \xrightarrow{\simeq} W$.
It is *trivial* if $W \cong M_0 \times [0,1]$ rel M_0.

The h-cobordism theorem ($n \geq 5$) [Smale]

If $\pi_1(M_i) = 0$ then any h-cobordism $(W; M_0, M_1)$ is trivial.

The *Whitehead group* of a group G is

$$\mathrm{Wh}(G) := K_1(\mathbb{Z}G)/(\pm G)$$

where $K_1(R) = \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)] = \mathrm{GL}(R)/E(R)$.

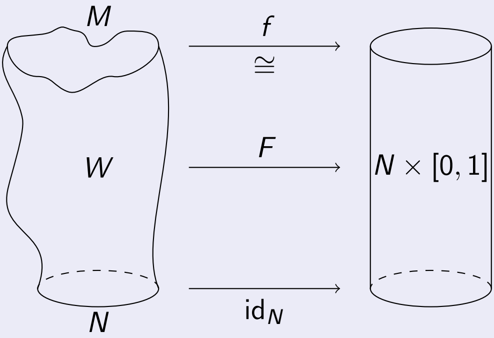
The s-cobordism theorem ($n \geq 5$) [Barden,Mazur,Stallings]

An h-cobordism $(W; M_0, M_1)$ with $G = \pi_1(M_0)$ is trivial if and only if

$$0 = \tau(W, M_0) \in \mathrm{Wh}(G).$$

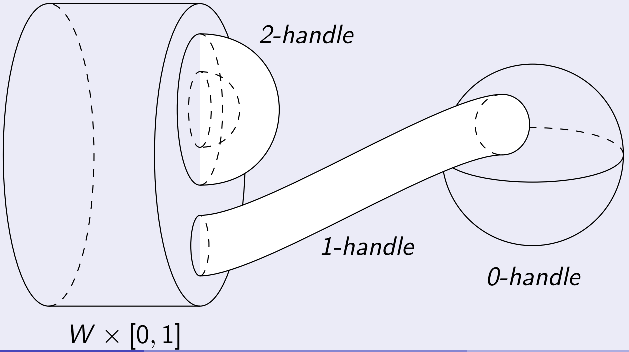
The s-cobordism theorem II

Figure (Trivial h-cobordism)



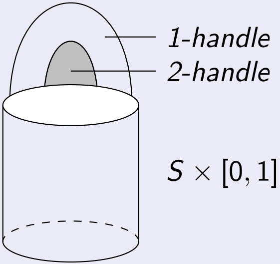
The *s*-cobordism theorem III

Figure (Handlebody decomposition)



The s-cobordism theorem IV

Figure (Handle cancellation)



The s-cobordism theorem V

Proof idea:

Reduce the handlebody decomposition to

$$W \;\cong\; \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}).$$

$C_*(\widetilde{W}, \widetilde{\partial_0 W})$ is acyclic.

The differential $d_{q+1} \colon H_{q+1}(\widetilde{W_{q+1}}, \widetilde{W_q}) \to H_q(\widetilde{W_q}, \widetilde{W_{q-1}})$ is bijective.

Let A represent d_{q+1} with respect to the bases $\{[\phi_i^{q+1}]\}$ and $\{[\phi_i^q]\}$.

The element $\tau(W, M_0)$ is defined by A.

Surgery theory

Basic idea

We want an h-cobordism $(W; M_0, M_1)$.

Step 1: Get any cobordism $(W; M_0, M_1)$.
Step 2: Improve W to be an h-cobordism, that means $W \xrightarrow{\simeq} M_0 \times [0, 1]$.

Technical assumptions:

Step 1: Get a normal cobordism $(W; M_0, M_1)$.
Step 2: Improve W to be an h-cobordism $W \xrightarrow{\simeq} M_0 \times [0, 1]$.

Absolute situation with X an n-dim geometric Poincaré complex:

Step 1: Get a normal map $(f, \overline{f}) \colon M \to X$ with M closed manifold.
Step 2: Improve $(f, \overline{f})$ to a homotopy equivalence $f' \colon M' \to X$.

Normal invariants I (Step 1)

Let X be a finite n-GPC.

A degree one normal map on X is

$$(f, \overline{f}) \colon M \to X$$

with M an n-mfd and $\overline{f} \colon \nu_M \to \xi$ a bundle map.

$$\begin{array}{ccc} \nu_M & \xrightarrow{\overline{f}} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow[f]{} & X \end{array}$$

Btw. $S(\xi) \simeq \nu_X$ where ν_X is the Spivak normal fibration.

Normal invariants II (Step 1)

Let X be a finite n-GPC.

Define

$$((f_0, \overline{f}_0)\colon M_0 \to X) \;\sim\; ((f_1, \overline{f}_1)\colon M_1 \to X)$$

if there exists

$$(F, \overline{F})\colon (W; M_0, M_1) \to (X \times [0,1], X \times 0, X \times 1)$$

such that for $j = 0, 1$

$$(F, \overline{F}) \circ i_j = (f_j, \overline{f}_j).$$

Definition:

The normal invariants of X is

$$\mathcal{N}^{\mathrm{TOP}}(X) := \{(f, \overline{f})\colon M \to X\}/\sim$$

$\mathcal{N}^{\mathrm{TOP}}(X)$ could be empty. It is in principle calculable (Pontrjagin-Thom).

Surgery obstructions I (Step 2)

Let $(f, \overline{f})\colon M \to X$ be a degree one normal map.
How to make $(f, \overline{f})$ into a homotopy equivalence?
Surgery:
Suppose we can represent an element $0 \neq \alpha \in \pi_k(f)$ by (q embedding):

$$\begin{array}{ccc} S^k \times D^{n-k} & \xrightarrow{q} & M \\ \downarrow j & & \downarrow f \\ D^{k+1} \times D^{n-k} & \xrightarrow[Q]{} & X \end{array}$$

Construct:

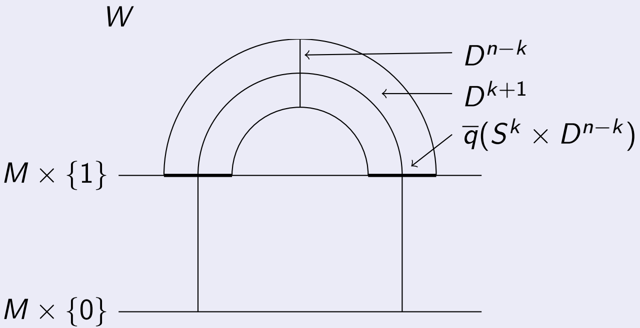
$$M' := D^{k+1} \times S^{n-k-1} \cup_{\mathrm{im}(q|_{S^k \times S^{n-k-1}})} (M - (\mathrm{im}(q)))\,.$$

$$W = D^{k+1} \times D^{n-k} \cup_q M \times [0,1],$$

This kills α.

Surgery obstructions II (Step 2)

Figure (normal bordism)



Surgery obstructions III (Step 2)

Surgery possibly changes $H_i(M)$ for $i = k, k+1$ and $n-k, n-k-1$. Therefore it works well for $2k+1 < n$.

For $2k = n$ and $2k+1 = n$ we get obstructions in general.

Case $2k = n$.

The L-group $L_{2k}(R)$ = iso classes of non-degenerate $(-1)^k$-quadratic forms modulo hyperbolic forms.

Theorem (surgery obstructions, $n \geq 5$, Wall)

There exists an element $\mathbf{sign}^{\mathbf{L}\bullet}(f, \overline{f}) \in L_{2k}(\mathbb{Z}G)$ such that $\mathbf{sign}^{\mathbf{L}\bullet}(f, \overline{f}) = 0$ iff $(f, \overline{f}) \sim (f', \overline{f}')$ with f' a homotopy equivalence.

$\mathbf{sign}^{\mathbf{L}\bullet} : \mathcal{N}^{\mathrm{TOP}}(X) \to L_n(\mathbb{Z}G)$

Surgery obstructions IV (Step 2)

An ε-quadratic form (P, ψ) is $\psi \in \text{coker}(1 - \varepsilon T \colon \text{Hom}_R(P, P^*))$.

A standard hyperbolic ε-quadratic form is $H_\varepsilon(L) = (L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$.

A form is called hyperbolic if it is a finite direct sum $H_\varepsilon(L)$.

The surgery obstruction

$$\mathbf{sign}^{\mathbf{L}^\bullet}(f, \overline{f}) = [(K_k(\widetilde{M}), \lambda, \mu)]$$

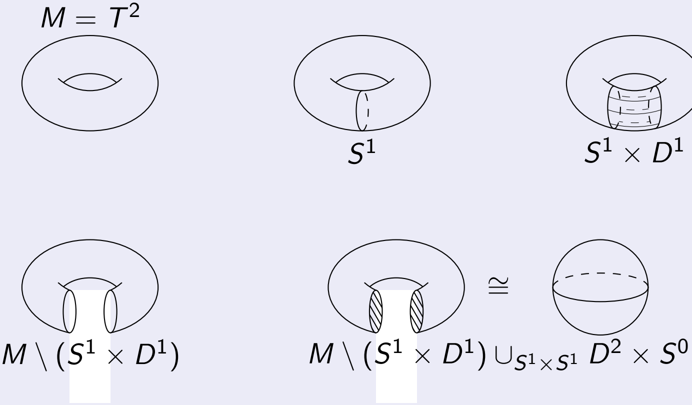
$K_k(\widetilde{M}) = \ker H_k(\widetilde{f}) \colon H_k(\widetilde{M}) \to H_k(\widetilde{X})$,

λ - intersection and μ - self-intersection numbers over $\mathbb{Z}G$.

Check $(f, \overline{f}) \colon S^k \times S^k \to S^{2k}$.

Surgery obstructions V (Step 2)

Figure (Product of spheres)



The surgery exact sequence

Let X be an n-GPC. Then we have:

The surgery exact sequence for X ($\dim(X) = n \geq 5$)

$$\cdots \mathcal{N}_{\partial}^{\mathrm{TOP}}(X \times I) \to L_{n+1}(\mathbb{Z}[\pi]) \xrightarrow{\partial} \mathcal{S}^{\mathrm{TOP},h}(X) \xrightarrow{\eta} \mathcal{N}^{\mathrm{TOP}}(X) \to L_n(\mathbb{Z}[\pi])$$

η - obvious

∂ - plumbing

$\mathcal{N}^{\mathrm{TOP}}(X) \cong [X; \mathrm{G/TOP}]$ is a combination of

$H^*(-;\mathbb{Z}_{(2)})$, $H^*(-;\mathbb{Z}/2)$ and $KO^*(-)[1/2]$.

$\pi_i(\mathrm{G/TOP}) = L_i(\mathbb{Z}) = \mathbb{Z}, 0, \mathbb{Z}_2, 0$

The proof of BC for $X = T^n$ is in this setup directly ([KS, 1977]).

Summary of the classical approach

Algebraic K-theory

$$\pm G \to K_1(\mathbb{Z}G) \to \mathrm{Wh}(G)$$

Surgery theory

$$\mathcal{S}^{\mathrm{TOP},h}(X) \to \mathcal{N}^{\mathrm{TOP}}(X) \to L_n(\mathbb{Z}[\pi])$$

K-theory and L-theory

K-theory

$K_n(R)$ for $n \neq 1$?

$K_n(R) = \pi_n B\mathrm{GL}(R)^+ \cong \pi_n B\mathcal{QP}(R) \cong \pi_n \Omega |w\mathcal{S}_\bullet \mathcal{P}(R)|$ for $n \geq 1$.

Here $\mathcal{P}(R)$ is the cat of f.g. projective R-modules.

One can define $K_n(R)$ also for $n \leq 0$.

L-theory

$L_n(R) =$ iso classes of non-degenerate $(-1)^k$-quadratic forms modulo hyperbolic forms ($n = 2k$).

$L_n(R) =$ iso classes of non-degenerate $(-1)^k$-quadratic formations modulo boundary and trivial formations ($n = 2k+1$).

$L_n(R) =$ cobordism groups of n-dimensional quadratic Poincaré chain complexes (Ranicki)

Spectra and homology theories

There exists a spectrum $\mathbf{K}(R)$ such that $\pi_i \mathbf{K}(R) = K_i(R)$.

There exists a spectrum $\mathbf{L}(R)$ such that $\pi_i \mathbf{L}(R) = L_i(R)$.

$\mathbf{L}(R)$ has versions with decorations.

A homology theory $H_* \colon \text{Spaces} \to \text{Ab}_*$ is a collection of functors satisfying homotopy invariance, long exact sequences, excision (and infinite sums).

For any homology theory H_* there is a spectrum $\mathbf{E}$ such that

$$H_n(X; \mathbf{E}) = \pi_n(X_+ \wedge \mathbf{E}).$$

The functors $X \to \pi_n \mathbf{K}(\mathbb{Z}[G])$ and $X \to \pi_n \mathbf{L}(\mathbb{Z}[G])$ are not homology theories.

Assembly maps I

For any homotopy functor F : Spaces $\to$ Spectra there exists a homotopy functor $F^{\%}$: Spaces $\to$ Spectra which is the best approximation of F by a homology theory from the left:

$$\mathrm{asmb}\colon F^{\%}(X) \to F(X)$$

$$F^{\%}(X) \simeq X_+ \wedge F(*)$$

We get

$$X_+ \wedge \mathbf{K}(\mathbb{Z}) \xrightarrow{\mathrm{asmb}} \mathbf{K}(\mathbb{Z}[G]) \to \mathbf{Wh}_{\mathrm{lin}}(X)$$

and

$$\mathbf{S}(X) \to X_+ \wedge \mathbf{L}(\mathbb{Z}) \xrightarrow{\mathrm{asmb}} \mathbf{L}(\mathbb{Z}[G])$$

Assembly maps II

We also have

$$\pi_1 \mathbf{Wh}_{\mathrm{lin}}(X) = \mathrm{Wh}(G) \quad \text{and} \quad \pi_0 \mathbf{S}(X) = \mathcal{S}^{\mathrm{TOP},h}(X)$$

The geometric surgery exact sequences $\rightsquigarrow$ algebraic surgery exact sequence

$$\begin{array}{ccccccc}
L_{n+1}(\mathbb{Z}[\pi]) & \xrightarrow{\partial} & \mathcal{S}^{\mathrm{TOP}}(X) & \xrightarrow{\eta} & \mathcal{N}^{\mathrm{TOP}}(X) & \xrightarrow{\mathbf{sign}_\pi} & L_n(\mathbb{Z}[\pi]) \\
\downarrow = & & \downarrow \mathbf{sign}_X & & \downarrow \mathbf{sign}_X & & \downarrow = \\
L_{n+1}(\mathbb{Z}[\pi]) & \xrightarrow{\partial} & \mathbb{S}_{n+1}(X) & \longrightarrow & H_n(X; \mathbf{L}_\bullet\langle 1 \rangle) & \xrightarrow{\mathrm{asmb}} & L_n(\mathbb{Z}[\pi])
\end{array}$$

where $\mathbb{S}_{n+1} = \pi_0 \mathbf{S}(X)$.

This uses algebraic theory of surgery due to Ranicki.

The Farrell-Jones conjecture I

In order to prove the Borel conjecture for a group G it suffices to show:

The Farrell-Jones conjecture (torsion-free case, over $\mathbb{Z}$)

If G is a torsion-free finitely presented group then

$$BG_+ \wedge \mathbf{K}(\mathbb{Z}) \xrightarrow{\text{asmb}} \mathbf{K}(\mathbb{Z}[G])$$

and

$$BG_+ \wedge \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z}) \xrightarrow{\text{asmb}} \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z}[G])$$

induce isomorphisms on all homotopy groups.

The Farrell-Jones conjecture II

The Farrell-Jones conjecture (general case, over $\mathbb{Z}$)

If G is a finitely presented group then

$$H_n^G(E_{\mathcal{VCYC}}G; \mathbf{K}(\mathbb{Z})) \xrightarrow{\text{asmb}} K_n(\mathbb{Z}[G])$$

and

$$H_n^G(E_{\mathcal{VCYC}}G; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \xrightarrow{\text{asmb}} L_n^{\langle -\infty \rangle}(\mathbb{Z}[G])$$

are isomorphisms.

Controlled topology I - thin h-cobordism theorem

Let M be a mfd with metric d, let $\varepsilon > 0$. An h-cobordism $(W; M, N)$ is called ε-controlled if $\exists\ p\colon W \to M$ and $H\colon \mathrm{id}_W \simeq p$ s.t. $\forall x \in W$

$$\mathrm{diam}\{p(H(x,t))\} < \varepsilon$$

Theorem [thin h-cobordism] (Quinn)

Let M be a mfd with $\dim(M) \geq 5$ and let d be a metric on M. There exists $\varepsilon > 0$ such that any ε-controlled h-cobordism is trivial.

Controlled topology II - algebraic thin h-cobordism theorem

Setup: X free G-space, Z simplicial complex $\dim(Z) = N$ with G-action, $p\colon X \to Z$ a G-map.

A geometric $R[G]$-module over X is $M = (M_x)_{x \in X}$ f.g free R-modules s.t.

- $M_x = M_{gx}$
- supp $M = G \cdot S_0$ for some $|S_0| < \infty$.

A morphism $f\colon \oplus M_x \to \oplus N_x$ is an $R[G]$-linear map. It is ε-controlled if $\forall (x'', x') \in \text{supp } f$

$$d_Z(p(x''), p(x')) < \varepsilon$$

Theorem [algebraic thin h-cobordism]

Given $N \geq 0$ there exists $\varepsilon > 0$ s.t. if $f\colon M \to M$ is an ε_N-controlled automorphism over Z then $0 = [f] \in \text{Wh}(G)$.

Controlled topology IV - flow methods

Setup:
M Riemannian manifold, TM tangent bundle, $\widetilde{M}$ universal cover,
SM the sphere bundle of the tangent bundle.
$v \in TM \quad \rightsquigarrow \quad \alpha_v : \mathbb{R} \to M$ geodesic
The geodesic flow

$$g : \mathbb{R} \times SM \to SM$$
$$g^t(v) = \dot{\alpha}_v(t)$$

Cartan-Hadamard: if M is not-positively curved then $\exp : T_{x_0}M \to M$ is the universal covering.
Compactification

$$\overline{M} = \widetilde{M} \cup M(\infty)$$
$$M(\infty) = \{\alpha_v(t) \mid \in [0, \infty)\} / \sim$$

Controlled topology V - flow methods

Farrell-Hsiang used $\overline{M}$ to show that if M is not-positively curved then the following holds:

Condition $(*)$

A closed manifold M^n satisfies $(*)$ if $\exists$ action $\alpha \colon G \times D^n \to D^n$ such that

1. the restriction of α to int D^n is equivalent to G-action on $\widetilde{M}$
2. $\forall$ compact $K \subset$ int D^n and $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall \gamma \in G$ we have

$$d(\gamma K, \partial D^n) < \delta \quad \Rightarrow \quad \mathrm{diam}(\gamma K) < \varepsilon.$$

This enable to obtain 'more control' and prove injectivity of asmb in L-theory.
Farrell-Jones developed this further using asymptotic foliation on SM and foliated thin h-cobordism theorem to show $\mathrm{Wh}(G) = 0$. Note that the flow is on SM and not M. This leads to transfers!

Controlled topology VI - other examples

Word-hyperbolic groups:

Let $\delta > 0$. A discrete group G is called δ-hyperbolic if any geodesic triangle in the Cayley graph of G is δ-thin with respect to the word metric. It is called word-hyperbolic if it is δ-hyperbolic for some $\delta > 0$.

The definition does not depend on the choice of the generating set for the Cayley graph.

Examples: finite groups, virtually cyclic groups, finitely generated free groups, and more generally, groups that act on a locally finite tree with finite stabilizers, the fundamental groups of surfaces with negative Euler characteristic, the fundamental groups of compact Riemannian manifolds with strictly negative sectional curvature, groups that act cocompactly and properly discontinuously on a proper $\mathrm{CAT}(k)$ space with $k < 0$.

Examples of non-hyperbolic groups: $\mathbb{Z}^2$, any group which contains $\mathbb{Z}^2$ as a subgroup, Baumslag-Solitar groups $B(m, n)$

Status of the Farrell-Jones conjecture I

Theorem (Bartels, Farrell, Kammeyer, Lück, Reich, Ruping, Wegner)

Let $\mathcal{FJ}$ be the class of groups for which the Full Farrell-Jones Conjecture holds. Then $\mathcal{FJ}$ contains the following groups:

- hyperbolic groups;
- CAT(0)-groups;
- solvable groups,
- (Not necessarily uniform) lattices in almost connected Lie groups;
- $\pi_1 M$ with M (not necessarily compact) d-dimensional manifolds (possibly with boundary) for $d \leq 3$.
- Subgroups of $\mathrm{GL}_n(\mathbb{Q})$ and of $\mathrm{GL}_n(F[t])$ for a finite field F
- S-arithmetic groups.

Status of the Farrell-Jones conjecture II

Theorem continued

Moreover, $\mathcal{FJ}$ has the following inheritance properties:

- If G_1 and G_2 belong to $\mathcal{FJ}$, then $G_1 \times G_2$ and $G_1 * G_2$ belong to $\mathcal{FJ}$;
- If H is a subgroup of G and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;
- If $H \subseteq G$ is a subgroup of G with $[G : H] < \infty$ and $H \in \mathcal{FJ}$, then $G \in \mathcal{FJ}$;
- Let $\{G_i \mid i \in I\}$ be a directed system of groups s.t. $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\mathrm{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;

Open cases

- mapping class groups;
- $\mathrm{Out}(F_n)$;
- amenable groups;
- Thompsons groups;
- $G = F_n \rtimes \mathbb{Z}$.

Concluding remarks

Modern proofs: geometric group theory and flow methods

Calculations and applications possible

Novikov conjecture, Baum-Connes conjecture

Existence: ANR homology manifolds and the Quinn obstruction